Problem Set 2

Due: **9:59pm, Monday, 6 February**

This problem set focuses on understanding uncountability (Chapter 2 in TCS) and introducing the Boolean circuit model for finite computation (Chapter 3 in TCS).

There is no Jupyter notebook part for this problem set. You should complete the assignment by writing your answers in the ps2.tex LaTeX template, similarly to how you did Problem Set 1, and will submit your solutions as a PDF file in GradeScope.

Collaboration Policy: You may discuss the problems with anyone you want. You are permitted to use any resources you find for this assignment **other than solutions from previous cs3102/cs3120 courses**. You should write up your own solutions and understand everything in them, and submit only your own work. You should note in the *Collaborators and Resources* box below the people you collaborated with and any external resources you used (you do not need to list resources you used for help with LaTeX or Jupyter/Python).

Collaborators and Resources: TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

To do the LaTeX part of this assignment, follow the directions from Problem Set 1. The URL for the template is: https://www.overleaf.com/read/rzzzgwrfmmwh

Before submitting your ps2.pdf file, also remember to:

– List your collaborators and resources, replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. (Remember to update this before submitting if you work with more people.)

– Replace the second line in ps2.tex, `\usepackage{uvatoc}` with `\usepackage[response]{uvatoc}` so the directions do not appear in your final PDF.

Problem 1 *Countable Graphs:* The book defines an *undirected graph* (Definition 1.3). We modify this by adding one word to define an *undirected finite graph* as follows:

Definition 1 (Undirected finite graph) An undirected finite graph $G = (V, E)$ consists of a *finite* set V of vertices and a set E of edges. Every edge is a size-two subset of V.

Prove that the set of all undirected finite graphs is *countably infinite*.

Answer:

Problem 2 *Uncountable Sets*

($\star$)[1] Prove that the set of all undirected graphs (using the book's Definition 1.3, without the constraint that V is finite that was added for the previous problem) is not countable. (Note: be careful in any argument that you make that the graphs you are counting are actually *different*, when we delete the labels of the nodes. Informally speaking, the two different graphs should have *different structure*. More formally, two graphs are called *isomorphic* if one can map their vertices using a bijection in such a way that this bijection leads to a bijection between the edges as well (i.e., if b is the bijection between vertices, then the mapping $b(\{i,j\}) = \{b(i), b(j)\}$ is a bijection between the edges of the two graphs). We say that two graphs have different structure if they are not isomorphic.)

Answer:

[1] When a problem is marked with a $\star$, it means we think this problem is challenging enough that students who are not able to solve it well can still get "full credit" for the assignment without submitting a perfect answer to the problem. We still hope everyone will attempt these problems and learn from trying to solve them, but you shouldn't get overly frustrated if you are not able to solve a $\star$ problem. This problem is here because it follows from the previous one, but we recommend completing the other problems in this assignment before returning to this one.

Problem 3 *Infinite Dominoes*

A domino is a tile with an unordered pair of numbers on it (e.g. $(0,5)$ or $(3,3)$). Dominoes come in sets containing all pairs of natural numbers less than or equal to some upper bound.[2]

A pack of "double 6" dominoes will contain all unordered pairs of values from the set $\{0,1,2,3,4,5,6\}$ (there will be 28 = 7 + 6 + 5 + 4 + 3 + 2 + 1 total). A pack of "double 3" dominoes will contain all unordered pairs of values from the set $\{0,1,2,3\}$ (there will be 10 total).

A *domino chain* is an ordered sequence of dominoes such that: (1) each domino's content is also ordered (e.g., $(5,0)$ is a different ordering tan $(0,5)$ for the same domino), and that (2) the second value of each domino matches the first value of the next. For example, the domino sequence $(1,2)(2,5)(5,5)(5,0)$ is a valid domino chain, whereas $(1,2)(2,5)(5,5)(0,0)$ is not.

Consider a pack of "double $\mathbb{N}$" dominoes, which contains all of the infinitely-many unordered pairs of natural numbers. Show that there is an uncountable number of infinite-length domino chains that can be constructed from a pack of "double $\mathbb{N}$" dominoes.

Answer:

[2] Note that we usually use $(\cdot,\cdot)$ to represent ordered pairs, but in this problem, the notation refers to unordered pairs that could have the same content repeated as well.

Problem 4 *Warming Cantorvanian Fingers*

The Cantorvanian creatures from the planet Cantorvania have only one hand, but it has a countably infinite number of fingers. A human glove has only 5 holes for fingers, so when a Cantorvanian wears one it will put many fingers into the same finger hole. To wear a glove, Cantorvanians do not need to put their fingers into the glove's holes contiguously. For example, fingers 4 and 7 may go into hole 2, with finger 5 going into hole 5. It is also okay if some glove holes have no fingers, but each of the fingers must be in exactly one hole.

Show that there is an uncountable number of ways for a Cantorvanian to wear a human glove.

Answer:

Problem 5 *Maximum number of Inputs*

The *depth* of a circuit is the length of the longest path (in the number of gates) from the an input to an output in the circuit. We say that an output y_j depends on an input x_i, if there are two input sequences that *only* differ in x_i and lead to different values in y_i. Suppose a Boolean circuit C has n inputs $x_1, \ldots, x_n$, has depth at most d for some $d \leq 0$, has only one output y, and that y depends on *all* of the n. Prove $n \leq 2^d$. (Note: there are different ways to prove this, but we recommend using induction.)

Answer:

Problem 6 *Compare 4 bit numbers* (Exercise 3.1 in TCS book)

Draw a Boolean circuit (using only *AND*, *OR*, and *NOT* gates) that computes the function $CMP_8 : \{0,1\}^8 \rightarrow \{0,1\}$ such that $CMP_8(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3) = 1$ if and only if the number represented by $a_0a_1a_2a_3$ is larger than the number represented by $b_0b_1b_2b_3$. We will say that a_0, b_0 are the most significant bits and a_3, b_3 are least significant.

Answer:

Problem 7 *Compare n bit numbers* (Exercise 3.2 in TCS book)

Prove that there exists a constant c such that for every n there is a Boolean circuit (using only *AND*, *OR*, and *NOT* gates) C of at most $c \cdot n$ gates that computes the function $CMP_{2n} : \{0,1\}^{2n} \to \{0,1\}$ such that $CMP_{2n}(a_0 \cdots a_{n-1} b_0 \cdots b_{n-1}) = 1$ if and only if the number represented by $a_0 \cdots a_{n-1}$ is larger than the number represented by $b_0 \cdots b_{n-1}$.

In other words, generalize the previous problem to describe how to compare n-bit numbers for any specific value n using *AND*, *OR*, and *NOT* in such a way that the total number of gates used is $O(n)$ (i.e., asymptotically linear).

