Learning Objectives:

- Learn the model of *Turing machines*, which can compute functions of *arbitrary input lengths*.
- See a programming-language description of Turing machines, using NAND-TM programs, which add *loops* and *arrays* to NAND-CIRC.
- See some basic syntactic sugar and equivalence of variants of Turing machines and NAND-TM programs.

6
Loops and infinity

"An algorithm is a finite answer to an infinite number of questions.", Attributed to Stephen Kleene.

"The bounds of arithmetic were however outstepped the moment the idea of applying the [punched] cards had occurred; and the Analytical Engine does not occupy common ground with mere"calculating machines."" ... In enabling mechanism to combine together general symbols, in successions of unlimited variety and extent, a uniting link is established between the operations of matter and the abstract mental processes of the most abstract branch of mathematical science. ", Ada Augusta, countess of Lovelace, 1843

The model of Boolean circuits (or equivalently, the NAND-CIRC programming language) has one very significant drawback: a Boolean circuit can only compute a *finite* function f, and in particular since every gate has two inputs, a size s circuit can compute on an input of length at most $2s$. This does not capture our intuitive notion of an algorithm as a *single recipe* to compute a potentially infinite function. For example, the standard elementary school multiplication algorithm is a *single* algorithm that multiplies numbers of all lengths, but yet we cannot express this algorithm as a single circuit, but rather need a different circuit (or equivalently, a NAND-CIRC program) for every input length (see Fig. 6.1).

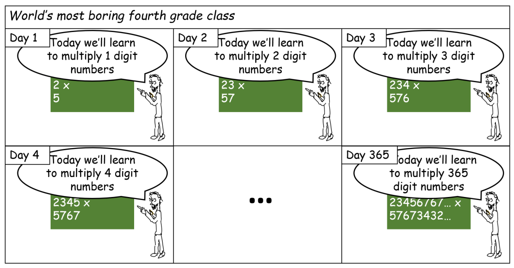
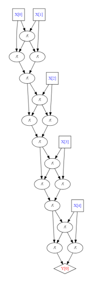

World's most boring fourth grade class
Day 1
Today we'll learn to multiply 1 digit numbers
2 x 5
Day 2
Today we'll learn to multiply 2 digit numbers
23 x 57
Day 3
Today we'll learn to multiply 3 digit numbers
234 x 576
Day 4
Today we'll learn to multiply 4 digit numbers
2345 x 5767
...
Day 365
Today we'll learn to multiply 365 digit numbers
23456767... x 57673432...

Figure 6.1: Once you know how to multiply multi-digit numbers, you can do so for every number n of digits, but if you had to describe multiplication using NAND-CIRC programs or Boolean circuits, you would need a different program/circuit for every length n of the input.

Let us consider the case of the simple *parity* or *XOR* function $XOR : \{0,1\}^* \rightarrow \{0,1\}$, where $XOR(x)$ equals 1 iff the number of 1's in x is odd. (In other words, $XOR(x) = \sum_{i=0}^{|x|-1} x_i \mod 2$ for every $x \in \{0,1\}^*$.) As simple as it is, the XOR function cannot be computed by a NAND-CIRC program. Rather, for every n, we can compute XOR_n (the restriction of XOR to $\{0,1\}^n$) using a different NAND-CIRC program. For example, Fig. 6.2 presents the NAND-CIRC program (or equivalently the circuit) to compute XOR_5.



```
Temp[0] = NAND(X[0],X[1])
Temp[1] = NAND(X[0],Temp[0])
Temp[2] = NAND(X[1],Temp[0])
Temp[3] = NAND(Temp[1],Temp[2])
Temp[4] = NAND(X[2],Temp[3])
Temp[5] = NAND(X[2],Temp[4])
Temp[6] = NAND(Temp[3],Temp[4])
Temp[7] = NAND(Temp[5],Temp[6])
Temp[8] = NAND(Temp[7],X[3])
Temp[9] = NAND(Temp[7],Temp[8])
Temp[10] = NAND(X[3],Temp[8])
Temp[11] = NAND(Temp[9],Temp[10])
Temp[12] = NAND(Temp[11],X[4])
Temp[13] = NAND(Temp[11],Temp[12])
Temp[14] = NAND(X[4],Temp[12])
Y[0] = NAND(Temp[13],Temp[14])
```

Figure 6.2: The NAND circuit and NAND-CIRC program for computing the XOR of 5 bits. Note how the circuit for XOR_5 merely repeats four times the circuit to compute the XOR of 2 bits.

This code for computing XOR_5 is rather repetitive, and more importantly, does not capture the fact that there is a *single* algorithm to compute the parity on all inputs. Typical programming language use the notion of *loops* to express such an algorithm, along the lines of:

```
# s is the "running parity", initialized to 0
while i<len(X):
    u = NAND(s,X[i])
    v = NAND(s,u)
    w = NAND(X[i],u)
    s = NAND(v,w)
    i+= 1
Y[0] = s
```

Generally an algorithm is, as we quote above, "a finite answer to an infinite number of questions". To express an algorithm we need to write down a finite set of instructions that will enable us to compute on arbitrarily long inputs. To describe and execute an algorithm we need the following components (see Fig. 6.3):

- The finite set of instructions to be performed.
- Some "local variables" or finite state used in the execution.
- A potentially unbounded working memory to store the input as well as any other values we may require later.
- While the memory is unbounded, at every single step we can only read and write to a finite part of it, and we need a way to *adress* which are the parts we want to read from and write to.
- If we only have a finite set of instructions but our input can be arbitrarily long, we will need to *repeat* instructions (i.e., *loop* back). We need a mechanism to decide when we will loop and when we will halt.

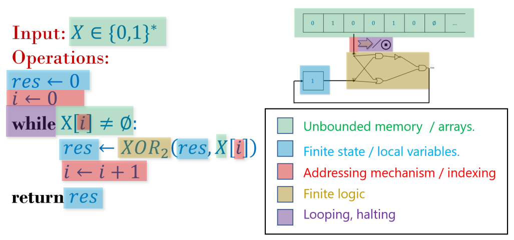
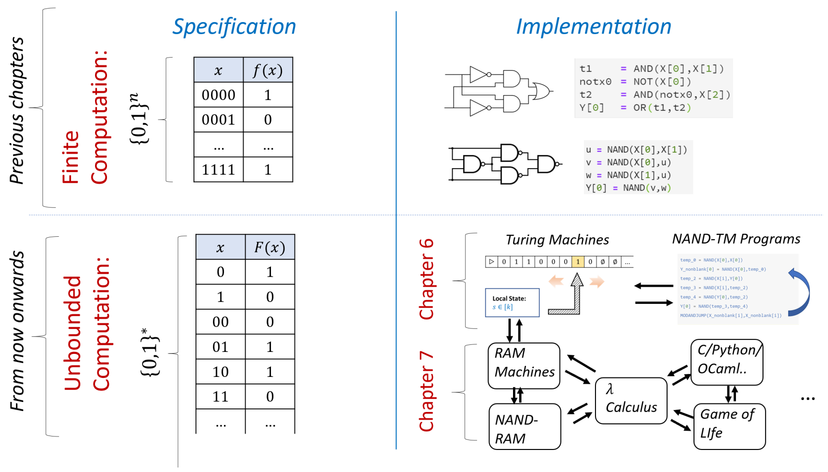

Input: $X \in \{0,1\}^*$
Operations:
$res \leftarrow 0$
$i \leftarrow 0$
while X[i] $\neq \emptyset$:
$res \leftarrow XOR_2(res, X[i])$
$i \leftarrow i+1$
return res
Unbounded memory / arrays.
Finite state / local variables.
Addressing mechanism / indexing
Finite logic
Looping, halting

Figure 6.3: An algorithm is a finite recipe to compute on arbitrarily long inputs. The components of an algorithm include the instructions to be performed, finite state or "local variables", the memory to store the input and intermediate computations, as well as mechanisms to decide which part of the memory to access, and when to repeat instructions and when to halt.

In this chapter we will show how we can extend the model of Boolean circuits / straight-line programs so that it can capture these kinds of constructs. We will see two ways to do so:

- *Turing machines*, invented by Alan Turing in 1936, are an hypothetical abstract device that yields a finite description of an algorithm that can handle arbitrarily long inputs.
- The *NAND-TM Programming language* extends NAND-CIRC with the notion of *loops* and *arrays* to obtain finite programs that can compute a function with arbitrarily long inputs.

It turns out that these two models are *equivalent*, and in fact they are equivalent to a great many other computational models including programming languages you may be familiar with such as C, Lisp, Python, JavaScript, etc. This notion, known as *Turing equivalence* or *Turing completeness*, will be discussed in Chapter 7. See Fig. 6.4 for an overview of the models presented in this chapter and Chapter 7.



```
t1     = AND(X[0],X[1])
notx0  = NOT(X[0])
t2     = AND(notx0,X[2])
Y[0]   = OR(t1,t2)
```

```
u = NAND(X[0],X[1])
v = NAND(X[0],u)
w = NAND(X[1],u)
Y[0] = NAND(v,w)
```

```
temp_0 = NAND(X[0],X[0])
Y_nonblank[0] = NAND(X[0],temp_0)
temp_2 = NAND(X[i],Y[0])
temp_3 = NAND(X[i],temp_2)
temp_4 = NAND(Y[0],temp_2)
Y[0] = NAND(temp_3,temp_4)
MODANDJUMP(X_nonblank[i],X_nonblank[i])
```

Figure 6.4: Overview of our models for finite and unbounded computation. In the previous chapters we study the computation of *finite functions*, which are functions $f : \{0,1\}^n \rightarrow \{0,1\}^m$ for some fixed n, m, and modeled computing these functions using circuits or straightline programs. In this chapter we study computing *unbounded* functions of the form $F : \{0,1\}^* \rightarrow \{0,1\}^m$ or $F : \{0,1\}^* \rightarrow \{0,1\}^*$. We model computing these functions using *Turing Machines* or (equivalently) NAND-TM programs which add the notion of *loops* to the NAND-CIRC programming language. In Chapter 7 we will show that these models are equivalent to many other models, including RAM machines, the λ calculus, and all the common programming languages including C, Python, Java, JavaScript, etc.

R

Remark 6.1 — Finite vs infinite computation. Previously in this book we studied the computation of *finite* functions $f : \{0,1\}^n \rightarrow \{0,1\}^m$. Such a function f can always be described by listing all the 2^n values it takes on inputs $x \in \{0,1\}^n$.

In this chapter we consider functions that take inputs of *unbounded* size, such as the function $XOR : \{0,1\}^* \rightarrow \{0,1\}$ that maps x to $\sum_{i=0}^{|x|-1} x_i$ mod 2. While we can describe XOR using a finite number of symbols (in fact we just did so in the previous sentence), it takes infinitely many possible inputs and so we cannot just write down all of its values. The same is true for many other functions capturing important computational tasks including addition, multiplication, sorting, finding paths in graphs, fitting curves to points, and so on and so forth.

To contrast with the finite case, we will sometimes call a function $F : \{0,1\}^* \rightarrow \{0,1\}$ (or $F : \{0,1\}^* \rightarrow \{0,1\}^*$) *infinite* but we emphasize that the functions we are interested in always take an input which is a finite string. It's just that, unlike the finite case, this string can be arbitrarily long and is not fixed to some particular length n.

Some texts present the task of computing a function $F : \{0,1\}^* \rightarrow \{0,1\}$ as the task of deciding membership in the *language* $L \subseteq \{0,1\}^*$ defined as $L = \{x \in \{0,1\}^* \mid F(x) = 1\}$. These two views are equivalent, see Remark 6.5.

6.1 TURING MACHINES

"Computing is normally done by writing certain symbols on paper. We may suppose that this paper is divided into squares like a child's arithmetic book.. The behavior of the [human] computer at any moment is determined by the symbols which he is observing, and of his 'state of mind' at that moment... We may suppose that in a simple operation not more than one symbol is altered.",

"We compare a man in the process of computing ... to a machine which is only capable of a finite number of configurations... The machine is supplied with a 'tape' (the analogue of paper) ... divided into sections (called 'squares') each capable of bearing a 'symbol' ", Alan Turing, 1936

"What is the difference between a Turing machine and the modern computer? It's the same as that between Hillary's ascent of Everest and the establishment of a Hilton hotel on its peak." , Alan Perlis, 1982.

The "granddaddy" of all models of computation is the *Turing Machine*. Turing machines were defined in 1936 by Alan Turing in an attempt to formally capture all the functions that can be computed by human "computers" (see Fig. 6.6) that follow a well-defined set of rules, such as the standard algorithms for addition or multiplication.

Turing thought of such a person as having access to as much "scratch paper" as they need. For simplicity we can think of this scratch paper as a one dimensional piece of graph paper (or *tape*, as it is commonly referred to), which is divided to "cells", where each "cell" can hold a single symbol (e.g., one digit or letter, and more generally some element of a finite *alphabet*). At any point in time, the person can read from and write to a single cell of the paper, and based on the contents can update his/her finite mental state, and/or move to the cell immediately to the left or right of the current one.

Turing modeled such a computation by a "machine" that maintains one of k states. At each point in time the machine read from its "work tape" a single symbol from a finite alphabet Σ and use that to update its state, write to tape, and possible move to an adjacent cell (see Fig. 6.9). To compute a function F using this machine, we initialize the tape with the input $x \in \{0,1\}^*$ and our goal is to ensure that the tape will contain the value $F(x)$ at the end of the computation. Specifically,

Figure 6.5: Aside from his many other achievements, Alan Turing was an excellent long distance runner who just fell shy of making England's olympic team. A fellow runner once asked him why he punished himself so much in training. Alan said "I have such a stressful job that the only way I can get it out of my mind is by running hard; it's the only way I can get some release."

Figure 6.6: Until the advent of electronic computers, the word "computer" was used to describe a person that performed calculations. Most of these "human computers" were women, and they were absolutely essential to many achievements including mapping the stars, breaking the Enigma cipher, and the NASA space mission; see also the bibliographical notes. Photo taken from from [Sob17].

Figure 6.7: Steam-powered Turing Machine mural, painted by CSE grad students at the University of Washington on the night before spring qualifying examinations, 1987. Image from https://www.cs.washington.edu/building/art/SPTM.

a computation of a Turing Machine M with k states and alphabet Σ on input $x \in \{0,1\}^*$ proceeds as follows:

- Initially the machine is at state 0 (known as the "starting state") and the tape is initialized to $\triangleright, x_0, \ldots, x_{n-1}, \varnothing, \varnothing, \ldots$. We use the symbol $\triangleright$ to denote the beginning of the tape, and the symbol $\varnothing$ to denote an empty cell. We will always assume that the alphabet Σ is a (potentially strict) superset of $\{\triangleright, \varnothing, 0, 1\}$.
- The location i to which the machine points to is set to 0.
- At each step, the machine reads the symbol $\sigma = T[i]$ that is in the i^{th} location of the tape, and based on this symbol and its state s decides on:
  - What symbol σ' to write on the tape
  - Whether to move **L**eft (i.e., $i \leftarrow i-1$), **R**ight (i.e., $i \leftarrow i+1$), **S**tay in place, or **H**alt the computation.
  - What is going to be the new state $s \in [k]$
- The set of rules the Turing machine follows is known as its *transition function*.
- When the machine halts then its output is obtained by reading off the tape from the second location (just after the $\triangleright$) onwards, stopping at the first point where the symbol is not 0 or 1.

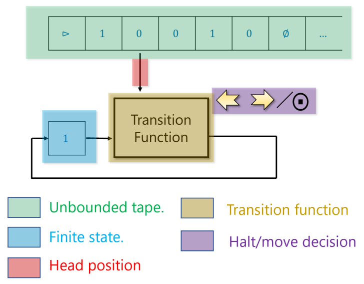


Figure 6.8: The components of a Turing Machine. Note how they correspond to the general components of algorithms as described in Fig. 6.3.

6.1.1 Extended example: A Turing machine for palindromes

Let PAL (for *palindromes*) be the function that on input $x \in \{0,1\}^*$, outputs 1 if and only if x is an (even length) *palindrome*, in the sense that $x = w_0 \cdots w_{n-1} w_{n-1} w_{n-2} \cdots w_0$ for some $n \in \mathbb{N}$ and $w \in \{0,1\}^n$.

We now show a Turing Machine M that computes PAL. To specify M we need to specify **(i)** M's tape alphabet Σ which should contain at least the symbols 0,1, $\triangleright$ and $\varnothing$, and **(ii)** M's *transition function* which determines what action M takes when it reads a given symbol while it is in a particular state.

In our case, M will use the alphabet $\{0, 1, \triangleright, \varnothing, \times\}$ and will have $k = 14$ states. Though the states are simply numbers between 0 and $k-1$, for convenience we will give them the following labels:

State	Label
0	`START`
1	`RIGHT_0`
2	`RIGHT_1`
3	`LOOK_FOR_0`

State	Label
4	`LOOK_FOR_1`
5	`RETURN`
6	`REJECT`
7	`ACCEPT`
8	`OUTPUT_0`
9	`OUTPUT_1`
10	`0_AND_BLANK`
11	`1_AND_BLANK`
12	`BLANK_AND_STOP`

We describe the operation of our Turing Machine M in words:

- M starts in state `START` and will go right, looking for the first symbol that is 0 or 1. If we find $\varnothing$ before we hit such a symbol then we will move to the `OUTPUT_1` state that we describe below.

- Once M finds such a symbol $b \in \{0, 1\}$, M deletes b from the tape by writing the $\times$ symbol, it enters either the `RIGHT_0` or `RIGHT_1` mode according to the value of b and starts moving rightwards until it hits the first $\varnothing$ or $\times$ symbol.

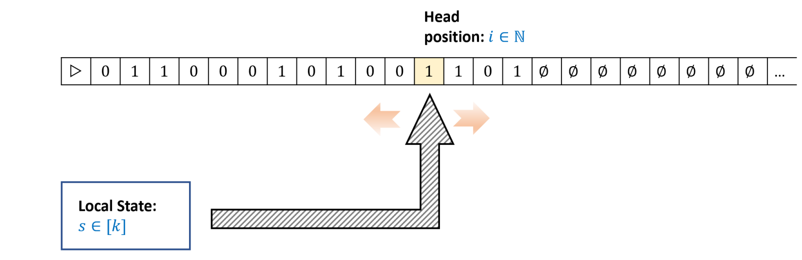
- Once we find this symbol we go into the state `LOOK_FOR_0` or `LOOK_FOR_1` depending on whether we were in the state `RIGHT_0` or `RIGHT_1` and make one left move.

- In the state `LOOK_FOR_`b, we check whether the value on the tape is b. If it is, then we delete it by changing its value to $\times$, and move to the state `RETURN`. Otherwise, we change to the `OUTPUT_0` state.

- The `RETURN` state means we go back to the beginning. Specifically, we move leftward until we hit the first symbol that is not 0 or 1, in which case we change our state to `START`.

- The `OUTPUT_`b states mean that we are going to output the value b. In both these states we go left until we hit $\triangleright$. Once we do so, we make a right step, and change to the `1_AND_BLANK` or `0_AND_BLANK` states respectively. In the latter states, we write the corresponding value, and then move right and change to the `BLANK_AND_STOP` state, in which we write $\varnothing$ to the tape and halt.

The above description can be turned into a table describing for each one of the $13 \cdot 5$ combination of state and symbol, what the Turing machine will do when it is in that state and it reads that symbol. This table is known as the *transition function* of the Turing machine.



Rules / transition function: $M : [k] \times \Sigma \to [k] \times \Sigma \times \{\mathbf{L}, \mathbf{R}, \mathbf{S}, \mathbf{H}\}$

```
If read 0 and state is 17 then change state to 29, write 1, and move left.

If read ∅ and state is 23 then change state to 15, write 0, and move right.

If read ▷ and state is 8  then change state to 12, write ▷, and halt.
 ...
```

Figure 6.9: A Turing machine has access to a *tape* of unbounded length. At each point in the execution, the machine can read a single symbol of the tape, and based on that and its current state, write a new symbol, update the tape, decide whether to move left, right, stay, or halt.

6.1.2 Turing machines: a formal definition

The formal definition of Turing machines is as follows:

Definition 6.2 — Turing Machine. A (one tape) *Turing machine* with k states and alphabet $\Sigma \supseteq \{0, 1, \triangleright, \varnothing\}$ is represented by a *transition function* $\delta_M : [k] \times \Sigma \to [k] \times \Sigma \times \{\mathsf{L}, \mathsf{R}, \mathsf{S}, \mathsf{H}\}$.

For every $x \in \{0,1\}^*$, the *output* of M on input x, denoted by $M(x)$, is the result of the following process:

- We initialize T to be the sequence $\triangleright, x_0, x_1, \ldots, x_{n-1}, \varnothing, \varnothing, \ldots$, where $n = |x|$. (That is, $T[0] = \triangleright$, $T[i+1] = x_i$ for $i \in [n]$, and $T[i] = \varnothing$ for $i > n$.)

- We also initialize $i = 0$ and $s = 0$.

- We then repeat the following process:

  1. Let $(s', \sigma', D) = \delta_M(s, T[i])$.
  2. Set $s \to s'$, $T[i] \to \sigma'$.
  3. If $D = \mathsf{R}$ then set $i \to i+1$, if $D = \mathsf{L}$ then set $i \to \max\{i-1, 0\}$. (If $D = \mathsf{S}$ then we keep i the same.)
  4. If $D = \mathsf{H}$ then halt.

- The *result* of the process, which we denote by $M(x)$, is the string $T[1], \ldots, T[m]$ where $m > 0$ is the smallest integer such that $T[m+1] \notin \{0,1\}$. If the process never ends then we write $M(x) = \perp$.

P

You should make sure you see why this formal definition corresponds to our informal description of a Turing Machine. To get more intuition on Turing Machines, you can explore some of the online available simulators such as Martin Ugarte's, Anthony Morphett's, or Paul Rendell's.

One should not confuse the *transition function* δ_M of a Turing machine M with the function that the machine computes. The transition function δ_M is a *finite* function, with $k|\Sigma|$ inputs and $4k|\Sigma|$ outputs. (Can you see why?) The machine can compute an *infinite* function F that takes as input a string $x \in \{0,1\}^*$ of arbitrary length and might also produce an arbitrary length string as output.

In our formal definition, we identified the machine M with its transition function δ_M since the transition function tells us everything we need to know about the Turing machine, and hence serves as a good mathematical representation of it. This choice of representation is somewhat arbitrary, and is based on our convention that the state space is always the numbers $\{0, \ldots, k-1\}$ with 0 as the starting state. Other texts use different conventions and so their mathematical definition of a Turing machine might look superficially different, but these definitions describe the same computational process and has the same computational powers. See Section 6.7 for a comparison between Definition 6.2 and the way Turing Machines are defined in texts such as Sipser [Sip97]. These definitions are equivalent despite their superficial differences.

6.1.3 Computable functions

We now turn to making one of the most important definitions in this book, that of *computable functions*.

Definition 6.3 — Computable functions. Let $F : \{0,1\}^* \rightarrow \{0,1\}^*$ be a (total) function and let M be a Turing machine. We say that M *computes* F if for every $x \in \{0,1\}^*$, $M(x) = F(x)$.

We say that a function F is *computable* if there exists a Turing machine M that computes it.

Defining a function "computable" if and only if it can be computed by a Turing machine might seem "reckless" but, as we'll see in Chapter 7, it turns out that being computable in the sense of Definition 6.3 is equivalent to being computable in essentially any reasonable model of computation. This is known as the *Church Turing Thesis*. (Unlike the *extended* Church Turing Thesis which we discussed in Section 5.6, the Church-Turing thesis itself is widely believed and there are no candidate devices that attack it.)

Big Idea 8 We can precisely define what it means for a function to be computable by *any possible algorithm*.

This is a good point to remind the reader that *functions* are *not* the same as *programs*:

$$\text{Functions} \neq \text{Programs} . \tag{6.1}$$

A Turing machine (or program) M can *compute* some function F, but it is not the same as F. In particular there can be more than one program to compute the same function. Being computable is a property of *functions*, not of machines.

We will often pay special attention to functions $F : \{0,1\}^* \rightarrow \{0,1\}$ that have a single bit of output. Hence we give a special name for the set of functions of this form that are computable.

Definition 6.4 — The class R. We define $\mathbf{R}$ be the set of all *computable* functions $F : \{0,1\}^* \rightarrow \{0,1\}$.

R

Remark 6.5 — Functions vs. languages. Many texts use the terminology of "languages" rather than functions to refer to computational tasks. The name "language" has its roots in *formal language theory* as pursued by linguists such as Noam Chomsky. A *formal language* is a subset $L \subseteq \{0,1\}^*$ (or more generally $L \subseteq \Sigma^*$ for some finite alphabet Σ). The *membership* or *decision* problem for a language L, is the task of determining, given $x \in \{0,1\}^*$, whether or not $x \in L$. A Turing machine M *decides* a language L if for every input $x \in \{0,1\}^*$, $M(x)$ outputs 1 if and only if $x \in L$. This is equivalent to computing the Boolean function $F : \{0,1\}^* \rightarrow \{0,1\}$ defined as $F(x) = 1$ iff $x \in L$. A language L is *decidable* if there is a Turing machine M that decides it. For historical reasons, some texts also call such a language *recursive* (which is the reason that the letter $\mathbf{R}$ is often used to denote the set of computable Boolean functions / decidable languages defined in Definition 6.4).

In this book we stick to the terminology of *functions* rather than languages, but all definitions and results can be easily translated back and forth by using the equivalence between the function $F : \{0,1\}^* \rightarrow \{0,1\}$ and the language $L = \{x \in \{0,1\}^* \mid F(x) = 1\}$.

6.1.4 Infinite loops and partial functions

One crucial difference between circuits/straight-line programs and Turing machines is the following. Looking at a NAND-CIRC program P, we can always tell how many inputs and how many outputs it has (by simply looking at the `X` and `Y` variables). Furthermore, we are guaranteed that if we invoke P on any input then *some* output will be produced.

In contrast, given any Turing machine M, we cannot determine a priori the length of the output. In fact, we don't even know if an output would be produced at all! For example, it is very easy to come up with a Turing machine whose transition function never outouts `H` and hence never halts.

If a machine M fails to stop and produce an output on some an input x, then it cannot compute any total function F, since clearly on input x, M will fail to output $F(x)$. However, P can still compute a *partial function*.[1]

[1] A *partial function* F from a set A to a set B is a function that is only defined on a *subset* of A, (see Section 1.4.3). We can also think of such a function as mapping A to $B \cup \{\perp\}$ where $\perp$ is a special "failure" symbol such that $F(a) = \perp$ indicates the function F is not defined on a.

For example, consider the partial function DIV that on input a pair (a, b) of natural numbers, outputs $\lceil a/b \rceil$ if $b > 0$, and is undefined otherwise. We can define a Turing machine M that computes DIV on input a, b by outputting the first $c = 0, 1, 2, \ldots$ such that $cb \geq a$. If $a > 0$ and $b = 0$ then the machine M will never halt, but this is OK, since DIV is undefined on such inputs. If $a = 0$ and $b = 0$, the machine M will output 0, which is also OK, since we don't care about what the program outputs on inputs on which DIV is undefined. Formally, we define computability of partial functions as follows:

Definition 6.6 — Computable (partial or total) functions. Let F be either a total or partial function mapping $\{0,1\}^*$ to $\{0,1\}^*$ and let M be a Turing machine. We say that M *computes* F if for every $x \in \{0,1\}^*$ on which F is defined, $M(x) = F(x)$. We say that a (partial or total) function F is *computable* if there is a Turing machine that computes it.

Note that if F is a total function, then it is defined on every $x \in \{0,1\}^*$ and hence in this case, Definition 6.6 is identical to Definition 6.3.

R

Remark 6.7 — Bot symbol. We often use $\perp$ as our special "failure symbol". If a Turing machine M fails to halt on some input $x \in \{0,1\}^*$ then we denote this by $M(x) = \perp$. This *does not* mean that M outputs some encoding of the symbol $\perp$ but rather that M enters into an infinite loop when given x as input.

If a partial function F is undefined on x then we can also write $F(x) = \perp$. Therefore one might think that Definition 6.6 can be simplified to requiring that $M(x) = F(x)$ for every $x \in \{0,1\}^*$, which would imply that for every x, M halts on x if and only if F is defined on x. However this is not the case: for a Turing Machine M to compute a partial function F it is not *necessary* for M to enter an infinite loop on inputs x on which F is not defined. All that is needed is for M to output $F(x)$ on x's on which F is defined: on other inputs it is OK for M to output an arbitrary value such as 0, 1, or anything else, or not to halt at all. To borrow a term from the `C` programming language, on inputs x on which F is not defined, what M does is "undefined behavior".

6.2 TURING MACHINES AS PROGRAMMING LANGUAGES

The name "Turing machine", with its "tape" and "head" evokes a physical object, while in contrast we think of a *program* as a piece of text. But we can think of a Turing machine as a program as well. For example, consider the Turing Machine M of Section 6.1.1 that computes the function PAL such that $PAL(x) = 1$ iff x is a palindrome. We can also describe this machine as a *program* using the Python-like pseudocode of the form below

```
# Gets an array Tape initialized to
# [">", x_0 , x_1 , .... , x_(n-1), " ", " ", ...]
# At the end of the execution, Tape[1] is equal to 1
# if x is a palindrome and is equal to 0 otherwise
def PAL(Tape):
    head = 0
    state = 0 # START
    while (state != 12):
        if (state == 0 && Tape[head]=='0'):
            state = 3 # LOOK_FOR_0
            Tape[head] = 'x'
            head += 1 # move right
        if (state==0 && Tape[head]=='1')
            state = 4 # LOOK_FOR_1
            Tape[head] = 'x'
            head += 1 # move right
        ... # more if statements here
```

The particular details of this program are not important. What matters is that we can describe Turing machines as *programs*. Moreover, note that when translating a Turing machine into a program, the *tape* becomes a *list* or *array* that can hold values from the finite set Σ.[2] The *head position* can be thought of as an integer valued variable that can hold integers of unbounded size. The *state* is a *local register* that can hold one of a fixed number of values in $[k]$.

More generally we can think of every Turing Machine M as equivalent to a program similar to the following:

```
# Gets an array Tape initialized to
# [">", x_0 , x_1 , .... , x_(n-1), " ", " ", ...]
def M(Tape):
    state = 0
    i     = 0 # holds head location
    while (True):
        # Move head, modify state, write to tape
        # based on current state and cell at head
        # below are just examples for how program looks
         ↪  for a particular transition function
        if Tape[i]=="0" and state==7: #
         ↪  δ_M(7,"0")=(19,"1","R")
            i += 1
            Tape[i]="1"
            state = 19
        elif Tape[i]==">" and state == 13: #
         ↪  δ_M(13,">")=(15,"0","S")
            Tape[i]="0"
            state = 15
        elif ...
        ...
        elif Tape[i]==">" and state == 29: #
         ↪  δ_M(29,">")=(.,.,"H")
            break # Halt
```

If we wanted to use only *Boolean* (i.e., $0/1$-valued) variables then we can encode the `state` variables using $\lceil \log k \rceil$ bits. Similarly, we can represent each element of the alphabet Σ using $\ell = \lceil \log |\Sigma| \rceil$ bits and hence we can replace the Σ-valued array `Tape[]` with ℓ Boolean-valued arrays `Tape0[]`,..., `Tape`ℓ`[]`.

6.2.1 The NAND-TM Programming language

We now introduce the *NAND-TM programming language*, which aims to capture the power of a Turing machine in a programming language formalism. Just like the difference between Boolean circuits and Turing Machines, the main difference between NAND-TM and NAND-CIRC is that NAND-TM models a *single uniform algorithm* that can compute a function that takes inputs of *arbitrary lengths*. To do so, we extend the NAND-CIRC programming language with two constructs:

- *Loops*: NAND-CIRC is a *straight-line* programming language- a NAND-CIRC program of s lines takes exactly s steps of computation and hence in particular cannot even touch more than $3s$ variables. *Loops* allow us to capture in a short program the instructions for a computation that can take an arbitrary amount of time.
- *Arrays*: A NAND-CIRC program of s lines touches at most $3s$ variables. While we can use variables with names such as `Foo_17` or `Bar[22]`, they are not true arrays, since the number in the identifier is a constant that is "hardwired" into the program.

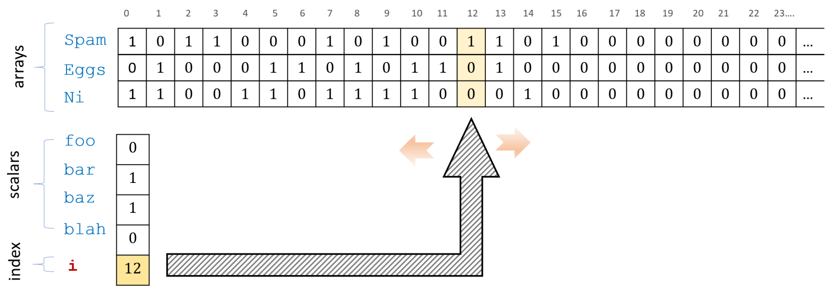


Figure 6.10: A NAND-TM program has *scalar* variables that can take a Boolean value, *array* variables that hold a sequence of Boolean values, and a special *index* variable `i` that can be used to index the array variables. We refer to the `i`-th value of the array variable `Spam` using `Spam[i]`. At each iteration of the program the index variable can be incremented or decremented by one step using the `MODANDJMP` operation.

Thus a good way to remember NAND-TM is using the following informal equation:

$$\text{NAND-TM} = \text{NAND-CIRC} + \text{loops} + \text{arrays} \qquad (6.2)$$

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Remark 6.8 — NAND-CIRC + loops + arrays = everything.. As we will see, adding loops and arrays to NAND-CIRC is enough to capture the full power of all programming languages! Hence we could replace "NAND-TM" with any of *Python*, *C*, *Javascript*, *OCaml*, etc. in the lefthand side of (6.2). But we're getting ahead of ourselves: this issue will be discussed in Chapter 7.

Concretely, the NAND-TM programming language adds the following features on top of NANC-CIRC (see Fig. 6.10)):

- We add a special *integer valued* variable `i`. All other variables in NAND-TM are *Boolean valued* (as in NAND-CIRC).

- Apart from `i` NAND-TM has two kinds of variables: *scalars* and *arrays*. *Scalar* variables hold one bit (just as in NAND-CIRC). *Array* variables hold an unbounded number of bits. At any point in the computation we can access the array variables at the location indexed by `i` using `Foo[i]`. We cannot access the arrays at locations other the one pointed to by `i`.

- We use the convention that *arrays* always start with a capital letter, and *scalar variables* (which are never indexed with `i`) start with lowercase letters. Hence `Foo` is an array and `bar` is a scalar variable.

- The input and output `X` and `Y` are now considered *arrays* with values of zeroes and ones. (There are also two other special arrays `X_nonblank` and `Y_nonblank`, see below.)

- We add a special `MODANDJUMP` instruction that takes two boolean variables a, b as input and does the following:

  - If $a = 1$ and $b = 1$ then `MODANDJUMP(`a, b`)` increments `i` by one and jumps to the first line of the program.
  - If $a = 0$ and $b = 1$ then `MODANDJUMP(`a, b`)` decrements `i` by one and jumps to the first line of the program. (If `i` is already equal to 0 then it stays at 0.)
  - If $a = 1$ and $b = 0$ then `MODANDJUMP(`a, b`)` jumps to the first line of the program without modifying `i`.
  - If $a = b = 0$ then `MODANDJUMP(`a, b`)` halts execution of the program.

- The`MODANDJUMP` instruction always appears in the last line of a NAND-TM program and nowhere else.

Default values. We need one more convention to handle "default values". Turing machines have the special symbol $\varnothing$ to indicate that tape location is "blank" or "uninitialized". In NAND-TM there is no such symbol, and all variables are *Boolean*, containing either 0 or 1. All variables and locations of arrays are default to 0 if they have not been initialized to another value. To keep track of whether a 0 in an array corresponds to a true zero or to an uninitialized cell, a programmer can always add to an array `Foo` a "companion array" `Foo_nonblank` and set `Foo_nonblank[i]` to 1 whenever the `i`'th location is initialized. In particular we will use this convention for the input and output arrays `X` and `Y`. A NAND-TM program has *four* special arrays `X`, `X_nonblank`, `Y`, and `Y_nonblank`. When a NAND-TM program is executed on input $x \in \{0,1\}^*$ of length n, the first n cells of the array `X` are initialized to $x_0, \ldots, x_{n-1}$ and the first n cells of the array `X_nonblank` are initialized to 1. (All uninitialized cells default to 0.) The output of a NAND-TM program is the string `Y[0]`, ..., `Y[`$m-1$`]` where m is the smallest integer such that `Y_nonblank[`m`]`$=0$. A NAND-TM program gets called with `X` and `X_nonblank` initialized to contain the input, and writes to `Y` and `Y_nonblank` to produce the output.

Formally, NAND-TM programs are defined as follows:

Definition 6.9 — NAND-TM programs. A *NAND-TM program* consists of a sequence of lines of the form `foo = NAND(bar,blah)` ending with a line of the form `MODANDJMP(foo,bar)`, where `foo`,`bar`,`blah` are either *scalar variables* (sequences of letters, digits, and underscores) or *array variables* of the form `Foo[i]` (starting with capital letter and indexed by `i`). The program has the array variables `X`, `X_nonblank`, `Y`, `Y_nonblank` and the index variable `i` built in, and can use additional array and scalar variables.

If P is a NAND-TM program and $x \in \{0,1\}^*$ is an input then an execution of P on x is the following process:

1. The arrays `X` and `X_nonblank` are initialized by `X[`i`]`$=x_i$ and `X_nonblank[`i`]`$=1$ for all $i \in [|x|]$. All other variables and cells are initialized to 0. The index variable `i` is also initalized to 0.

2. The program is executed line by line, when the last line `MODANDJMP(foo,bar)` is executed then we do as follows:

   a. If `foo`$=1$ and `bar`$=0$ then jump to the first line without modifying the value of `i`.

   b. If `foo`$=1$ and `bar`$=1$ then increment `i` by one and jump to the first line.

   c. If `foo`$=0$ and `bar`$=1$ then decrement `i` by one (unless it is already zero) and jump to the first line.

   d. If `foo`$=0$ and `bar`$=0$ then halt and output `Y[0]`, ..., `Y[`$m-1$`]` where m is the smallest integer such that `Y_nonblank[`m`]`$=0$.

6.2.2 Sneak peak: NAND-TM vs Turing machines

As the name implies, NAND-TM programs are a direct implementation of Turing machines in programming language form. we will show the equivalence below but you can already see how the components of Turing machines and NAND-TM programs correspond to one another:

Table 6.2: Turing Machine and NAND-TM analogs

Turing Machine	**NAND-TM program**
State: single register that takes values in $[k]$	*Scalar variables:* Several variables such as `foo`, `bar` etc.. each taking values in $\{0,1\}$.
Tape: One tape containing values in a finite set Σ. Potentially infinite but $T[t]$ defaults to $\varnothing$ for all locations t that have not been accessed.	*Arrays:* Several arrays such as `Foo`, `Bar` etc.. for each such array `Arr` and index j, the value of `Arr` at position j is either 0 or 1. The value defaults to 0 for position that have not been written to.
Head location: A number $i \in \mathbb{N}$ that encodes the position of the head.	*Index variable:* The variable `i` that can be used to access the arrays.
Accessing memory: At every step the Turing machine has access to its local state, but can only access the tape at the position of the current head location.	*Accessing memory:* At every step a NAND-TM program has access to all the scalar variables, but can only access the arrays at the location `i` of the index variable
Control of location: In each step the machine can move the head location by at most one position.	*Control of index variable:* In each iteration of its main loop the program can modify the index `i` by at most one.

6.2.3 Examples

We now present some examples of NAND-TM programs

■ **Example 6.10 — XOR in NAND-TM.** The following is a NAND-TM program to compute the XOR function on inputs of arbitrary length. That is $XOR : \{0,1\}^* \rightarrow \{0,1\}$ such that $XOR(x) = \sum_{i=0}^{|x|-1} x_i \mod 2$ for every $x \in \{0,1\}^*$.

```
temp_0 = NAND(X[0],X[0])
Y_nonblank[0] = NAND(X[0],temp_0)
temp_2 = NAND(X[i],Y[0])
temp_3 = NAND(X[i],temp_2)
temp_4 = NAND(Y[0],temp_2)
Y[0] = NAND(temp_3,temp_4)
MODANDJUMP(X_nonblank[i],X_nonblank[i])
```

■ **Example 6.11 — Increment in NAND-TM.** We now present NAND-TM program to compute the *increment function*. That is, $INC : \{0,1\}^* \rightarrow \{0,1\}^*$ such that for every $x \in \{0,1\}^n$, $INC(x)$ is the $n+1$ bit long string y such that if $X = \sum_{i=0}^{n-1} x_i \cdot 2^i$ is the number represented by x, then y is the (least-significant digit first) binary representation of the number $X+1$.

We start by showing the program using the "syntactic sugar" we've seen before of using shorthand for some NAND-CIRC programs we have seen before to compute simple functions such as IF, XOR and AND (as well as the constant one function as well as the function COPY that just maps a bit to itself).

```
carry = IF(started,carry,one(started))
started = one(started)
Y[i] = XOR(X[i],carry)
carry = AND(X[i],carry)
Y_nonblank[i] = one(started)
MODANDJUMP(X_nonblank[i],X_nonblank[i])
```

The above is not, strictly speaking, a valid NAND-TM program. If we "open up" all of the syntactic sugar, we get the following valid program to compute this syntactic sugar.

```
temp_0 = NAND(started,started)
temp_1 = NAND(started,temp_0)
temp_2 = NAND(started,started)
temp_3 = NAND(temp_1,temp_2)
temp_4 = NAND(carry,started)
carry = NAND(temp_3,temp_4)
temp_6 = NAND(started,started)
started = NAND(started,temp_6)
temp_8 = NAND(X[i],carry)
temp_9 = NAND(X[i],temp_8)
temp_10 = NAND(carry,temp_8)
Y[i] = NAND(temp_9,temp_10)
temp_12 = NAND(X[i],carry)
carry = NAND(temp_12,temp_12)
temp_14 = NAND(started,started)
Y_nonblank[i] = NAND(started,temp_14)
MODANDJUMP(X_nonblank[i],X_nonblank[i])
```

Working out the above two example can go a long way towards understanding the NAND-TM language. See the appendix and our GitHub repository for a full specification of the NAND-TM language.

6.3 EQUIVALENCE OF TURING MACHINES AND NAND-TM PROGRAMS

Given the above discussion, it might not be surprising that Turing machines turn out to be equivalent to NAND-TM programs. Indeed, we designed the NAND-TM language to have this property. Nevertheless, this is an important result, and the first of many other such equivalence results we will see in this book.

Theorem 6.12 — Turing machines and NAND-TM programs are equivalent. For every $F : \{0,1\}^* \rightarrow \{0,1\}^*$, F is computable by a NAND-TM program P if and only if there is a Turing Machine M that computes F.

Proof Idea:

To prove such an equivalence theorem, we need to show two directions. We need to be able to $\mathbf{(1)}$ transform a Turing machine M to a NAND-TM program P that computes the same function as M and $\mathbf{(2)}$ transform a NAND-TM program P into a Turing machine M that computes the same function as P.

The idea of the proof is illustrated in Fig. 6.11. To show $\mathbf{(1)}$, given a Turing machine M, we will create a NAND-TM program P that will have an array `Tape` for the tape of M and scalar (i.e., non array) variable(s) `state` for the state of M. Specifically, since the state of a Turing machine is not in $\{0,1\}$ but rather in a larger set $[k]$, we will use $\lceil \log k \rceil$ variables `state_`0 , ..., `state_`$\lceil \log k \rceil - 1$ variables to store the representation of the state. Similarly, to encode the larger alphabet Σ of the tape, we will use $\lceil \log |\Sigma| \rceil$ arrays `Tape_`0 , ..., `Tape_`$\lceil \log |\Sigma| \rceil - 1$, such that the i^{th} location of these arrays encodes the i^{th} symbol in the tape for every tape. Using the fact that *every* function can be computed by a NAND-CIRC program, we will be able to compute the transition function of M, replacing moving left and right by decrementing and incrementing `i` respectively.

We show $\mathbf{(2)}$ using very similar ideas. Given a program P that uses a array variables and b scalar variables, we will create a Turing machine with about 2^b states to encode the values of scalar variables, and an alphabet of about 2^a so we can encode the arrays using our tape. (The reason the sizes are only "about" 2^a and 2^b is that we will need to add some symbols and steps for bookkeeping purposes.) The

Turing Machine M will simulate each iteration of the program P by updating its state and tape accordingly.

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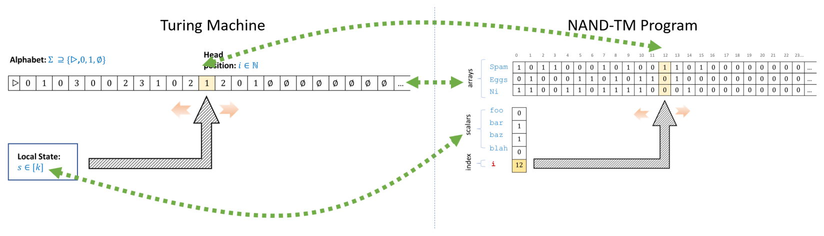

Figure 6.11: Comparing a Turing Machine to a NAND-TM program. Both have an unbounded memory component (the *tape* for a Turing machine, and the *arrays* for a NAND-TM program), as well as a constant local memory (*state* for a Turing machine, and *scalar variables* for a NAND-TM program). Both can only access at each step one location of the unbounded memory, this is the "head" location for a Turing machine, and the value of the index variable `i` for a NAND-TM program.

Proof of Theorem 6.12. We start by proving the "if" direction of Theorem 6.12. Namely we show that given a Turing machine M, we can find a NAND-TM program P_M such that for every input x, if M halts on input x with output y then $P_M(x) = y$. Since our goal is just to show such a program P_M *exists*, we don't need to write out the full code of P_M line by line, and can take advantage of our various "syntactic sugar" in describing it.

The key observation is that by Theorem 4.12 we can compute *every* finite function using a NAND-CIRC program. In particular, consider the transition function $\delta_M : [k] \times \Sigma \rightarrow [k] \times \Sigma \times \{\mathsf{L}, \mathsf{R}\}$ of our Turing Machine. We can encode the its components as follows:

- We encode $[k]$ using $\{0,1\}^{\ell}$ and Σ using $\{0,1\}^{\ell'}$, where $\ell = \lceil \log k \rceil$ and $\ell' = \lceil \log |\Sigma| \rceil$.

- We encode the set $\{\mathsf{L}, \mathsf{R}, \mathsf{S}, \mathsf{H}\}$ using $\{0,1\}^2$. We will choose the encode $\mathsf{L} \mapsto 01$, $\mathsf{R} \mapsto 11$, $\mathsf{S} \mapsto 10$, $\mathsf{H} \mapsto 00$. (This conveniently corresponds to the semantics of the `MODANDJUMP` operation.)

Hence we can identify δ_M with a function $\overline{M} : \{0,1\}^{\ell+\ell'} \rightarrow \{0,1\}^{\ell+\ell'+2}$, mapping strings of length $\ell + \ell'$ to strings of length $\ell + \ell' + 2$. By Theorem 4.12 there exists a finite length NAND-CIRC program `ComputeM` that computes this function $\overline{M}$. The NAND-TM program to simulate M will essentially be the following:

Algorithm 6.13 — NAND-TM program to simulate TM M.

```
Input: $x \in \{0,1\}^*$
Output: $M(x)$ if $M$ halts on $x$. Otherwise go into infinite loop
1: # We use variables state_0 ... state_$\ell - 1$ to encode $M$'s state
2: # We use arrays Tape_0[] ... Tape_$\ell' - 1$[] to encode $M$'s tape
3:         # We omit the initial and final "book keeping" to copy
Input: to Tape and copy
Output: from Tape
4:       # Use the fact that transition is finite and computable by NAND-CIRC program:
5: state_0 ... state_$\ell - 1$, Tape_0[i]... Tape_$\ell' - 1$[i], dir0,dir1 ← TRANSITION( state_0 ... state_$\ell - 1$, Tape_0[i]... Tape_$\ell' - 1$[i], dir0,dir1 )
6: MODANDJMP(dir0,dir1)
```

Every step of the main loop of the above program perfectly mimics the computation of the Turing Machine M and so the program carries out exactly the definition of computation by a Turing Machine as per Definition 6.2.

For the other direction, suppose that P is a NAND-TM program with s lines, ℓ scalar variables, and ℓ' array variables. We will show that there exists a Turing machine M_P with $2^\ell + C$ states and alphabet Σ of size $C' + 2^{\ell'}$ that computes the same functions as P (where C, C' are some constants to be determined later).

Specifically, consider the function $\overline{P} : \{0,1\}^\ell \times \{0,1\}^{\ell'} \rightarrow \{0,1\}^\ell \times \{0,1\}^{\ell'}$ that on input the contents of P's scalar variables and the contents of the array variables at location `i` in the beginning of an iteration, outputs all the new values of these variables at the last line of the iteration, right before the `MODANDJUMP` instruction is executed.

If `foo` and `bar` are the two variables that are used as input to the `MODANDJUMP` instruction, then this means that based on the values of these variables we can compute whether `i` will increase, decrease or stay the same, and whether the program will halt or jump back to the beginning. Hence a Turing machine can simulate an execution of P in one iteration using a finite function applied to its alphabet. The overall operation of the Turing machine will be as follows:

1. The machine M_P encodes the contents of the array variables of P in its tape, and the contents of the scalar variables in (part of) its state. Specifically, if P has ℓ local variables and t arrays, then the state space of M will be large enough to encode all 2^ℓ assignments

to the local variables and the alphabet Σ of M will be large enough to encode all 2^t assignments for the array variables at each location. The head location corresponds to the index variable `i`.

2. Recall that every line of the program P corresponds to reading and writing either a scalar variable, or an array variable at the location `i`. In one iteration of P the value of `i` remains fixed, and so the machine M can simulate this iteration by reading the values of all array variables at `i` (which are encoded by the single symbol in the alphabet Σ located at the `i`-th cell of the tape) , reading the values of all scalar variables (which are encoded by the state), and updating both. The transition function of M can output `L`, `S`, `R` depending on whether the values given to the `MODANDJMP` operation are 01, 10 or 11 respectively.

3. When the program halts (i.e., `MODANDJMP` gets 00) then the Turing machine will enter into a special loop to copy the results of the `Y` array into the output and then halt. We can achieve this by adding a few more states.

The above is not a full formal description of a Turing Machine, but our goal is just to show that such a machine exists. One can see that M_P simulates every step of P, and hence computes the same function as P.

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Remark 6.14 — Running time equivalence (optional). If we examine the proof of Theorem 6.12 then we can see that the every iteration of the loop of a NAND-TM program corresponds to one step in the execution of the Turing machine. We will come back to this question of measuring number of computation steps later in this course. For now the main take away point is that NAND-TM programs and Turing Machines are essentially equivalent in power even when taking running time into account.

6.3.1 Specification vs implementation (again)

Once you understand the definitions of both NAND-TM programs and Turing Machines, Theorem 6.12 is fairly straightforward. Indeed, NAND-TM programs are not as much a different model from Turing Machines as they are simply a reformulation of the same model using programming language notation. You can think of the difference between a Turing machine and a NAND-TM program as the difference between representing a number using decimal or binary notation. In contrast, the difference between a *function* F and a Turing machine that computes F is much more profound: it is like the difference between the equation $x^2 + x = 12$ and the number 3 that is a solution for this equation. For this reason, while we take special care in distinguishing *functions* from *programs* or *machines*, we will often identify the two latter concepts. We will move freely between describing an algorithm as a Turing machine or as a NAND-TM program (as well as some of the other equivalent computational models we will see in Chapter 7 and beyond).

Table 6.3: Specification vs Implementation formalisms

Setting	*Specification*	*Implementation*
Finite computation	**Functions** mapping $\{0,1\}^n$ to $\{0,1\}^m$	**Circuits**, **Straightline programs**
Infinite computation	**Functions** mapping $\{0,1\}^*$ to $\{0,1\}$ or to $\{0,1\}^*$.	**Algorithms**, **Turing Machines**, **Programs**

6.4 NAND-TM SYNTACTIC SUGAR

Just like we did with NAND-CIRC in Chapter 4, we can use "syntactic sugar" to make NAND-TM programs easier to write. For starters, we can use all of the syntactic sugar of NAND-CIRC, and so have access to macro definitions and conditionals (i.e., if/then). But we can go beyond this and achieve for example:

- Inner loops such as the `while` and `for` operations commong to many programming language.s
- Multiple index variables (e.g., not just `i` but we can add `j`, `k`, etc.).
- Arrays with more than one dimension (e.g., `Foo[i][j]`, `Bar[i][j][k]` etc.)

In all of these cases (and many others) we can implement the new feature as mere "syntactic sugar" on top of standard NAND-TM, which means that the set of functions computable by NAND-TM with this feature is the same as the set of functions computable by standard NAND-TM. Similarly, we can show that the set of functions computable by Turing Machines that have more than one tape, or tapes of more dimensions than one, is the same as the set of functions computable by standard Turing machines.

6.4.1 "GOTO" and inner loops

We can implement more advanced *looping constructs* than the simple MODANDJUMP. For example, we can implement GOTO. A GOTO statement corresponds to jumping to a certain line in the execution. For example, if we have code of the form

```
"start":  do foo
   GOTO("end")
"skip": do bar
"end": do blah
```

then the program will only do foo and blah as when it reaches the line GOTO("end") it will jump to the line labeled with "end". We can achieve the effect of GOTO in NAND-TM using conditionals. In the code below, we assume that we have a variable pc that can take strings of some constant length. This can be encoded using a finite number of Boolean variables pc_0, pc_1, ..., pc_$k-1$, and so when we write below pc = "label" what we mean is something like pc_0 = 0,pc_1 = 1, ... (where the bits $0, 1, \ldots$ correspond to the encoding of the finite string "label" as a string of length k). We also assume that we have access to conditional (i.e., if statements), which we can emulate using syntactic sugar in the same way as we did in NAND-CIRC.

To emulate a GOTO statement, we will first modify a program P of the form

```
do foo
do bar
do blah
```

to have the following form (using syntactic sugar for if):

```
pc = "line1"
if (pc=="line1"):
    do foo
    pc = "line2"
if (pc=="line2"):
    do bar
    pc = "line3"
if (pc=="line3"):
    do blah
```

These two programs do the same thing. The variable pc corresponds to the "program counter" and tells the program which line to execute next. We can see that if we wanted to emulate a GOTO("line3") then we could simply modify the instruction pc = "line2" to be pc = "line3".

In NAND-CIRC we could only have `GOTO`s that go forward in the code, but since in NAND-TM everything is encompassed within a large outer loop, we can use the same ideas to implement `GOTO`'s that can go backwards, as well as conditional loops.

Other loops. Once we have `GOTO`, we can emulate all the standard loop constructs such as `while`, `do .. until` or `for` in NAND-TM as well. For example, we can replace the code

```
while foo:
    do blah
do bar
```

with

```
"loop":
    if NOT(foo): GOTO("next")
    do blah
    GOTO("loop")
"next":
    do bar
```

Remark 6.15 — GOTO's in programming languages. The `GOTO` statement was a staple of most early programming languages, but has largely fallen out of favor and is not included in many modern languages such as *Python*, *Java*, *Javascript*. In 1968, Edsger Dijkstra wrote a famous letter titled "Go to statement considered harmful." (see also Fig. 6.12). The main trouble with `GOTO` is that it makes analysis of programs more difficult by making it harder to argue about *invariants* of the program.
When a program contains a loop of the form:

```
for j in range(100):
    do something

do blah
```

you know that the line of code `do blah` can only be reached if the loop ended, in which case you know that `j` is equal to 100, and might also be able to argue other properties of the state of the program. In contrast, if the program might jump to `do blah` from any other point in the code, then it's very hard for you as the programmer to know what you can rely upon in this code. As Dijkstra said, such invariants are important because *"our intellectual powers are rather geared to master static relations and .. our powers to visualize processes evolving in time are relatively poorly developed"*

and so *"we should ... do ...our utmost best to shorten the conceptual gap between the static program and the dynamic process."*
That said, `GOTO` is still a major part of lower level languages where it is used to implement higher level looping constructs such as `while` and `for` loops. For example, even though *Java* doesn't have a `GOTO` statement, the Java Bytecode (which is a lower level representation of Java) does have such a statement. Similarly, Python bytecode has instructions such as `POP_JUMP_IF_TRUE` that implement the `GOTO` functionality, and similar instructions are included in many assembly languages. The way we use `GOTO` to implement a higher level functionality in NAND-TM is reminiscent of the way these various jump instructions are used to implement higher level looping constructs.

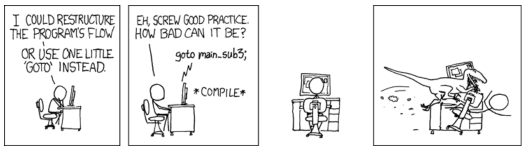
I COULD RESTRUCTURE THE PROGRAM'S FLOW
OR USE ONE LITTLE 'GOTO' INSTEAD.
EH, SCREW GOOD PRACTICE. HOW BAD CAN IT BE?
goto main_sub3;
COMPILE

Figure 6.12: XKCD's take on the `GOTO` statement.

6.5 UNIFORMITY, AND NAND VS NAND-TM (DISCUSSION)

While NAND-TM adds extra operations over NAND-CIRC, it is not exactly accurate to say that NAND-TM programs or Turing machines are "more powerful" than NAND-CIRC programs or Boolean circuits. NAND-CIRC programs, having no loops, are simply not applicable for computing functions with an unbounded number of inputs. Thus, to compute a function $F : \{0,1\}^* :\rightarrow \{0,1\}^*$ using NAND-CIRC (or equivalently, Boolean circuits) we need a *collection* of programs/circuits: one for every input length.

The key difference between NAND-CIRC and NAND-TM is that NAND-TM allows us to express the fact that the algorithm for computing parities of length-100 strings is really the same one as the algorithm for computing parities of length-5 strings (or similarly the fact that the algorithm for adding n-bit numbers is the same for every n, etc.). That is, one can think of the NAND-TM program for general parity as the "seed" out of which we can grow NAND-CIRC programs for length 10, length 100, or length 1000 parities as needed.

This notion of a single algorithm that can compute functions of all input lengths is known as *uniformity* of computation and hence we think of Turing machines / NAND-TM as *uniform* model of computation, as opposed to Boolean circuits or NAND-CIRC which is a *nonuniform* model, where we have to specify a different program for every input length.

Looking ahead, we will see that this uniformity leads to another crucial difference between Turing machines and circuits. Turing machines can have inputs and outputs that are longer than the description of the machine as a string and in particular there exists a Turing machine that can "self replicate" in the sense that it can print its own code. This notion of “self replication”, and the related notion of “self reference” is crucial to many aspects of computation, as well of course to life itself, whether in the form of digital or biological programs.

For now, what you ought to remember is the following differences between *uniform* and *non uniform* computational models:

- **Non uniform computational models:** Examples are *NAND-CIRC programs* and *Boolean circuits*. These are models where each individual program/circuit can compute a *finite* function $f : \{0,1\}^n \rightarrow \{0,1\}^m$. We have seen that *every* finite function can be computed by *some* program/circuit. To discuss computation of an *infinite* function $F : \{0,1\}^* \rightarrow \{0,1\}^*$ we need to allow a *sequence* $\{P_n\}_{n\in\mathbb{N}}$ of programs/circuits (one for every input length), but this does not capture the notion of a *single algorithm* to compute the function F.
- **Uniform computational models:** Examples are *Turing machines* and *NAND-TM programs*. These are model where a single program/machine can take inputs of *arbitrary length* and hence compute an *infinite* function $F : \{0,1\}^* \rightarrow \{0,1\}^*$. The number of steps that a program/machine takes on some input is not a priori bounded in advance and in particular there is a chance that it will enter into an *infinite loop*. Unlike the nonuniform case, we have *not* shown that every infinite function can be computed by some NAND-TM program/Turing Machine. We will come back to this point in Chapter 8.

✓ **Lecture Recap**

- *Turing machines* capture the notion of a single algorithm that can evaluate functions of every input length.
- They are equivalent to *NAND-TM programs*, which add loops and arrays to NAND-CIRC.
- Unlike NAND-CIRC or Boolean circuits, the number of steps that a Turing machine takes on a given input is not fixed in advance. In fact, a Turing machine or a NAND-TM program can enter into an *infinite loop* on certain inputs, and not halt at all.

6.6 EXERCISES

Exercise 6.1 — Explicit NAND TM programming. Produce the code of a (syntactic-sugar free) NAND-TM program P that computes the (unbounded input length) *Majority* function $Maj : \{0,1\}^* \rightarrow \{0,1\}$ where for every $x \in \{0,1\}^*$, $Maj(x) = 1$ if and only if $\sum_{i=0}^{|x|} x_i > |x|/2$. We say “produce” rather than “write” because you do not have to write the code of P by hand, but rather can use the programming language of your choice to compute this code.

■

Exercise 6.2 — Computable functions examples. Prove that the following functions are computable. For all of these functions, you do not have to fully specify the Turing Machine or the NAND-TM program that computes the function, but rather only prove that such a machine or program exists:

1. $INC : \{0,1\}^* \rightarrow \{0,1\}$ which takes as input a representation of a natural number n and outputs the representation of $n+1$.

2. $ADD : \{0,1\}^* \rightarrow \{0,1\}$ which takes as input a representation of a pair of natural numbers (n,m) and outputs the representation of $n+m$.

3. $MULT : \{0,1\}^* \rightarrow \{0,1\}^*$, which takes a representation of a pair of natural numbers (n,m) and outputs the representation of $n\dot{m}$.

4. $SORT : \{0,1\}^* \rightarrow \{0,1\}^*$ which takes as input the representation of a list of natural numbers $(a_0,\ldots,a_{n-1})$ and returns its sorted version $(b_0,\ldots,b_{n-1})$ such that for every $i \in [n]$ there is some $j \in [n]$ with $b_i = a_j$ and $b_0 \leq b_1 \leq \cdots \leq b_{n-1}$.

■

Exercise 6.3 — Two index NAND-TM. Define NAND-TM' to be the variant of NAND-TM where there are *two* index variables `i` and `j`. Arrays can be indexed by either `i` or `j`. The operation `MODANDJMP` takes four variables a,b,c,d and uses the values of c,d to decide whether to increment `j`, decrement `j` or keep it in the same value (corresponding to 01, 10, and 00 respectively). Prove that for every function $F : \{0,1\}^* \rightarrow \{0,1\}^*$, F is computable by a NAND-TM program if and only if F is computable by a NAND-TM' program.

■

Exercise 6.4 — Two tape Turing machines. Define a *two tape Turing machine* to be a Turing machine which has two separate tapes and two separate heads. At every step, the transition function gets as input the location of the cells in the two tapes, and can decide whether to move each head independently. Prove that for every function $F : \{0,1\}^* \rightarrow \{0,1\}^*$, F is computable by a standard Turing Machine if and only if F is computable by a two-tape Turing machine.

■

Exercise 6.5 — Two dimensional arrays. Define NAND-TM'' to be the variant of NAND-TM where just like NAND-TM' defined in Exercise 6.3

there are two index variables `i` and `j`, but now the arrays are *two dimensional* and so we index an array `Foo` by `Foo[i][j]`. Prove that for every function $F : \{0,1\}^* \rightarrow \{0,1\}^*$, F is computable by a NAND-TM program if and only if F is computable by a NAND-TM'' program.

■

Exercise 6.6 — Two tape Turing machines. Define a *two-dimensional Turing machine* to be a Turing machine in which the tape is *two dimensional*. At every step the machine can move Up, Down, Left, Right, or Stay. Prove that for every function $F : \{0,1\}^* \rightarrow \{0,1\}^*$, F is computable by a standard Turing Machine if and only if F is computable by a two-dimensional Turing machine.

■

Exercise 6.7 Prove the following closure properties of the set $\mathbf{R}$ defined in Definition 6.4:

1. If $F \in \mathbf{R}$ then the function $G(x) = 1 - F(x)$ is in $\mathbf{R}$.

2. If $F, G \in \mathbf{R}$ then the function $H(x) = F(x) \vee G(x)$ is in $\mathbf{R}$.

3. If $F \in \mathbf{R}$ then the function F^* in in $\mathbf{R}$ where F^* is defined as follows: $F^*(x) = 1$ iff there exist some strings $w_0, \ldots, w_{k-1}$ such that $x = w_0 w_1 \cdots w_{k-1}$ and $F(w_i) = 1$ for every $i \in [k]$.

4. If $F \in \mathbf{R}$ then the function

$$G(x) = \begin{cases} \exists_{y \in \{0,1\}^{|x|}} F(xy) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{6.3}$$

   is in $\mathbf{R}$.

■

Exercise 6.8 — Oblivious Turing Machines (challenging). Define a Turing Machine M to be *oblivious* if its head movement are independent of its input. That is, we say that M is oblivious if there existe an infinite sequence $MOVE \in \{\mathsf{L}, \mathsf{R}, \mathsf{S}\}^\infty$ such that for every $x \in \{0,1\}^*$, the movements of M when given input x (up until the point it halts, if such point exists) are given by $MOVE_0, MOVE_1, MOVE_2, \ldots$.

Prove that for every function $F : \{0,1\}^* \rightarrow \{0,1\}^*$, if F is computable then it is computable by an oblivious Turing machine. See footnote for hint.[3]

[3] You can use the sequence R, L,R, R, L, L, R,R,R, L, L, L, ….

■

Exercise 6.9 — Single vs multiple bit. Prove that for every $F : \{0,1\}^* \rightarrow \{0,1\}^*$, the function F is computable if and only if the following function $G : \{0,1\}^* \rightarrow \{0,1\}$ is computable, where G is defined as follows:

$$G(x,i,\sigma) = \begin{cases} F(x)_i & i < |F(x)|, \sigma = 0 \\ 1 & i < |F(x)|, \sigma = 1 \\ 0 & i \geq |F(x)| \end{cases}$$

■

Exercise 6.10 — Uncomputability via counting. Recall that **R** is the set of all total functions from $\{0,1\}^*$ to $\{0,1\}$ that are computable by a Turing machine (see Definition 6.4). Prove that **R** is *countable*. That is, prove that there exists a one-to-one map $DtN : \mathbf{R} \rightarrow \mathbb{N}$. You can use the equivalence between Turing machines and NAND-TM programs.

■

Exercise 6.11 — Not every function is computable. Prove that the set of *all* total functions from $\{0,1\}^* \rightarrow \{0,1\}$ is *not* countable. You can use the results of Section 2.3.1. (We will see an *explicit* uncomputable function in Chapter 8.)

■

6.7 BIBLIOGRAPHICAL NOTES

Augusta Ada Byron, countess of Lovelace (1815-1852) lived a short but turbulent life, though is today most well known for her collaboration with Charles Babbage (see [Ste87] for a biography). Ada took an immense interest in Babbage's *analytical engine*, which we mentioned in Chapter 3. In 1842-3, she translated from Italian a paper of Menabrea on the engine, adding copious notes (longer than the paper itself). The quote in the chapter's beginning is taken from Nota A in this text. Lovelace's notes contain several examples of *programs* for the analytical engine, and because of this she has been called "the world's first computer programmer" though it is not clear whether they were written by Lovelace or Babbage himself [Hol01]. Regardless, Ada was clearly one of very few people (perhaps the only one outside of Babbage himself) to fully appreciate how significant and revolutionary the idea of mechanizing computation truly is.

The books of Shetterly [She16] and Sobel [Sob17] discuss the history of human computers (who were female, more often than not) and their important contributions to scientific discoveries in astronomy and space exploration.

Alan Turing was one of the intellectual giants of the 20th century. He was not only the first person to define the notion of computation, but also invented and used some of the world's earliest computational devices as part of the effort to break the *Enigma* cipher during World War II, saving millions of lives. Tragically, Turing committed suicide in 1954, following his conviction in 1952 for homosexual acts and a court-mandated hormonal treatment. In 2009, British prime minister

Gordon Brown made an official public apology to Turing, and in 2013 Queen Elizabeth II granted Turing a posthumous pardon. Turing's life is the subject of a great book and a mediocre movie.

Sipser's text [Sip97] defines a Turing machine as a *seven tuple* consisting of the state space, input alphabet, tape alphabet, transition function, starting state, accpeting state, and rejecting state. Superficially this looks like a very different definition than Definition 6.2 but it is simply a different representation of the same concept, just as a graph can be represented in either adjacency list or adjacency matrix form.

One difference is that Sipser considers a general set of states Q that is not necessarily of the form $Q = \{0, 1, 2, \ldots, k-1\}$ for some natural number $k > 0$. Sipser also restricts his attention to Turing machines that output only a single bit and therefore designates two special *halting states*: the "0 halting state" (often known as the *rejecting state*) and the other as the "1 halting state" (often known as the *accepting state*). Thus instead of writing 0 or 1 on an output tape, the machine will enter into one of these states and halt. This again makes no difference to the computational power, though we prefer to consider the more general model of multi-bit outputs. (Sipser presents the basic task of a Turing machine as that of *deciding a language* as opposed to computing a function, but these are equivalent, see Remark 6.5.)

Sipser considers also functions with input in Σ^* for an arbitrary alphabet Σ (and hence distinguishes between the *input alphabet* which he denotes as Σ and the *tape alphabet* which he denotes as Γ), while we restrict attention to functions with binary strings as input. Again this is not a major issue, since we can always encode an element of Σ using a binary string of length $\log\lceil|\Sigma|\rceil$. Finally (and this is a very minor point) Sipser requires the machine to either move left or right in every step, without the Stay operation, though staying in place is very easy to emulate by simply moving right and then back left.

Another definition used in the literature is that a Turing machine M *recognizes* a language L if for every $x \in L$, $M(x) = 1$ and for every $x \notin L$, $M(x) \in \{0, \perp\}$. A language L is *recursively enumerable* if there exists a Turing machine M that recognizes it, and the set of all recursively enumerable languages is often denoted by **RE**. We will not use this terminology in this book.

One of the first programming-language formulations of Turing machines was given by Wang [Wan57]. Our formulation of NAND-TM is aimed at making the connection with circuits more direct, with the eventual goal of using it for the Cook-Levin Theorem, as well as results such as $\mathbf{P} \subseteq \mathbf{P}_{/\mathbf{poly}}$ and $\mathbf{BPP} \subseteq \mathbf{P}_{/\mathbf{poly}}$. The website esolangs.org features a large variety of esoteric Turing-complete programming languages. One of the most famous of them is Brainf*ck.