**Learning Objectives:**

- Introduce the notion of *polynomial-time reductions* as a way to relate the complexity of problems to one another.
- See several examples of such reductions.
- 3SAT as a basic starting point for reductions.

# 13 Polynomial-time reductions

Consider some of the problems we have encountered in Chapter 11:

1. The *3SAT* problem: deciding whether a given 3CNF formula has a satisfying assignment.
2. Finding the *longest path* in a graph.
3. Finding the *maximum cut* in a graph.
4. Solving *quadratic equations* over $n$ variables $x_0, \ldots, x_{n-1} \in \mathbb{R}$.

All of these problems have the following properties:

- These are important problems, and people have spent significant effort on trying to find better algorithms for them.
- Each one of these is a *search* problem, whereby we search for a solution that is "good" in some easy to define sense (e.g., a long path, a satisfying assignment, etc.).
- Each of these problems has a trivial exponential time algorithm that involve enumerating all possible solutions.
- At the moment, for all these problems the best known algorithm is not much faster than the trivial one in the worst case.

In this chapter and in Chapter 14 we will see that, despite their apparent differences, we can relate the computational complexity of these and many other problems. In fact, it turns out that the problem above are *computationally equivalent*, in the sense that solving one of them immediately implies solving the others. This phenomenon, known as **NP** *completeness*, is one of the surprising discoveries of theoretical computer science, and we will see that it has far-reaching ramifications.

In this chapter we will see that for each one of the problems of finding a longest path in a graph, solving quadratic equations, and finding

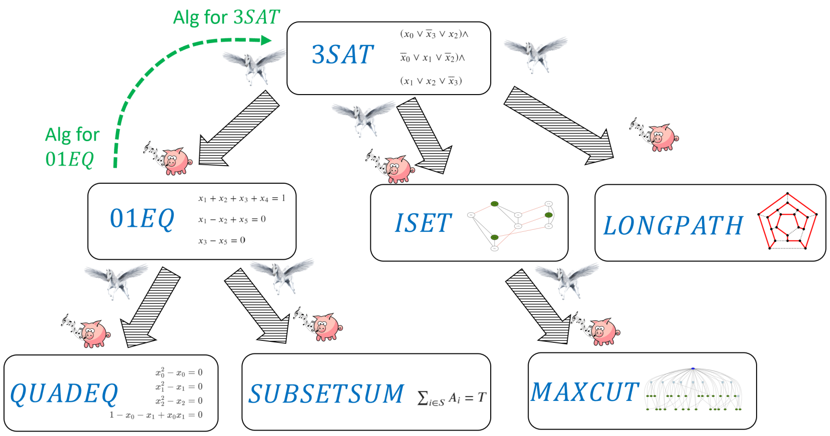


**Figure 13.1**: In this chapter we show that if the $3SAT$ problem cannot be solved in polynomial time, then neither can the $QUADEQ$, $LONGESTPATH$, $ISET$ and $MAXCUT$ problems. We do this by using the *reduction paradigm* showing for example "if pigs could whistle" (i.e., if we had an efficient algorithm for $QUADEQ$) then "horses could fly" (i.e., we would have an efficient algorithm for $3SAT$.)

the maximum cut, if there exists a polynomial-time algorithm for this problem then there exists a polynomial-time algorithm for the 3SAT problem as well. In other words, we will *reduce* the task of solving 3SAT to each one of the above tasks. Another way to interpret these results is that if there *does not exist* a polynomial-time algorithm for 3SAT then there does not exist a polynomial-time algorithm for these other problems as well. In Chapter 14 we will see evidence (though no proof!) that all of the above problems do not have polynomial-time algorithms and hence are *inherently intractable*.

## 13.1 FORMAL DEFINITIONS OF PROBLEMS

For reasons of technical convenience rather than anything substantial, we concern ourselves with *decision problems* (i.e., Yes/No questions) or in other words *Boolean* (i.e., one-bit output) functions. We model the problems above as functions mapping $\{0,1\}^*$ to $\{0,1\}$ in the following way:

**3SAT.** The *3SAT problem* can be phrased as the function $3SAT : \{0,1\}^* \rightarrow \{0,1\}$ that takes as input a 3CNF formula $\varphi$ (i.e., a formula of the form $C_0 \wedge \cdots \wedge C_{m-1}$ where each $C_i$ is the OR of three variables or their negation) and maps $\varphi$ to $1$ if there exists some assignment to the variables of $\varphi$ that causes it to evalute to *true*, and to $0$ otherwise. For example

$$3SAT\left("(x_0 \vee \overline{x}_1 \vee x_2) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x}_0 \vee \overline{x}_2 \vee x_3)"\right) = 1 \quad (13.1)$$

since the assignment $x = 1101$ satisfies the input formula. In the above we assume some representation of formulas as strings, and define the function to output 0 if its input is not a valid representation; we use the same convention for all the other functions below.

**Quadratic equations.** The *quadratic equations problem* corresponds to the function $QUADEQ : \{0,1\}^* \rightarrow \{0,1\}$ that maps a set of quadratic equations $E$ to 1 if there is an assignment $x$ that satisfies all equations, and to 0 otherwise.

**Longest path.** The *longest path problem* corresponds to the function $LONGPATH : \{0,1\}^* \rightarrow \{0,1\}$ that maps a graph $G$ and a number $k$ to 1 if there is a simple path in $G$ of length at least $k$, and maps $(G,k)$ to 0 otherwise. The longest path problem is a generalization of the well-known Hamiltonian Path Problem of determining whether a path of length $n$ exists in a given $n$ vertex graph.

**Maximum cut.** The *maximum cut problem* corresponds to the function $MAXCUT : \{0,1\}^* \rightarrow \{0,1\}$ that maps a graph $G$ and a number $k$ to 1 if there is a cut in $G$ that cuts at least $k$ edges, and maps $(G,k)$ to 0 otherwise.

All of the problems above are in **EXP** but it is not known whether or not they are in **P**. However, we will see in this chapter that if either $QUADEQ$ , $LONGPATH$ or $MAXCUT$ are in **P**, then so is $3SAT$.

## 13.2 POLYNOMIAL-TIME REDUCTIONS

Suppose that $F,G : \{0,1\}^* \rightarrow \{0,1\}$ are two functions. A *polynomial-time reduction* (or sometimes just *"reduction"* for short) from $F$ to $G$ is a way to show that $F$ is "no harder" than $G$, in the sense that a polynomial-time algorithm for $G$ implies a polynomial-time algorithm for $F$.

**Definition 13.1 — Polynomial-time reductions.** Let $F,G : \{0,1\}^* \rightarrow \{0,1\}^*$. We say that $F$ *reduces to* $G$, denoted by $F \leq_p G$ if there is a polynomial-time computable $R : \{0,1\}^* \rightarrow \{0,1\}^*$ such that for every $x \in \{0,1\}^*$,

$$F(x) = G(R(x)) \ . \tag{13.2}$$

We say that $F$ and $G$ have *equivalent complexity* if $F \leq_p G$ and $G \leq_p F$.

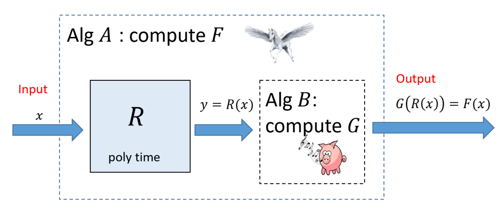


**Figure 13.2**: If $F \leq_p G$ then we can transform a polynomial-time algorithm $B$ that computes $G$ into a polynomial-time algorithm $A$ that computes $F$. To compute $F(x)$ we can run the reduction $R$ guaranteed by the fact that $F \leq_p G$ to obtain $y = F(x)$ and then run our algorithm $B$ for $G$ to compute $G(y)$.

The following exercise justifies our intuition that $F \leq_p G$ signifies that "$F$ is no harder than $G$.

**Solved Exercise 13.1 — Reductions and P.** Prove that if $F \leq_p G$ and $G \in$ **P** then $F \in$ **P**.

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P

As usual, solving this exercise on your own is an excellent way to make sure you understand Definition 13.1.

**Solution:**

Suppose there was an algorithm $B$ that compute $F$ in time $p(n)$ where $p$ is its input size. Then, (13.2) directly gives an algorithm $A$ to compute $F$ (see Fig. 13.2). Indeed, on input $x \in \{0,1\}^*$, Algorithm $A$ will run the polynomial-time reduction $R$ to obtain $y = R(x)$ and then return $B(y)$. By (13.2), $G(R(x)) = F(x)$ and hence Algorithm $A$ will indeed compute $F$.

We now show that $A$ runs in polynomial time. By assumption, $R$ can be computed in time $q(n)$ for some polynomial $q$. In particular, this means that $|y| \leq q(|x|)$ (as just writing down $y$ takes $|y|$ steps). This, computing $B(y)$ will take at most $p(|y|) \leq p(q(|x|))$ steps. Thus the total running time of $A$ on inputs of length $n$ is at most the time to compute $y$, which is bounded by $q(n)$, and the time to compute $B(y)$, which is bounded by $p(q(n))$, and since the composition of two polynomials is a polynomial, $A$ runs in polynomial time.

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A reduction from $F$ to $G$ can be used for two purposes:

- If we already know an algorithm for $G$ and $F \leq_p G$ then we can use the reduction to obtain an algorithm for $F$. This is a widely used tool in algorithm design. For example in Section 11.1.4 we saw how the *Min-Cut Max-Flow* theorem allows to reduce the task of computing a minimum cut in a graph to the task of computing a maximum flow in it.

- If we have proven (or have evidence) that there exists *no polynomial-time algorithm* for $F$ and $F \leq_p G$ then the existence of this reduction allows us to concludes that there exists no polynomial-time algorithm for $G$. This is the "if pigs could whistle then horses could fly" interpretation we've seen in Section 8.4. We show that if there was an hypothetical efficient algorithm for $G$ (a "whistling pig") then since $F \leq_p G$ then there would be an efficient algorithm for $F$ (a "flying horse"). In this book we often use reductions for this second purpose, although the lines between the two is sometimes blurry (see the bibliographical notes in Section 13.8).

The most crucial difference between the notion in Definition 13.1 and the reductions we saw in the context of *uncomputability* (e.g., in Section 8.4) is that for relating time complexity of problems, we need the reduction to be computable in *polynomial time*, as opposed to merely computable. Definition 13.1 also restricts reductions to have a very specific format. That is, to show that $F \leq_p G$, rather than allowing a general algorithm for $F$ that uses a "magic box" that computes $G$, we only allow an algorithm that computes $F(x)$ by outputting $G(R(x))$. This restricted form is convenient for us, but people have defined and used more general reductions as well (see Section 13.8).

In this chapter we use reductions to relate the computational complexity of the problems mentioned above: 3SAT, Quadratic Equations, Maximum Cut, and Longest Path, as well as a few others. We will reduce 3SAT to the latter problems, demonstrating that solving any one of them efficiently will result in an efficient algorithm for 3SAT. In Chapter 14 we show the other direction: reducing each one of these problems to 3SAT in one fell swoop.

**Transitivity of reductions.** Since we think of $F \leq_p G$ as saying that (as far as polynomial-time computation is concerned) $F$ is "easier or equal in difficulty to" $G$, we would expect that if $F \leq_p G$ and $G \leq_p H$, then it would hold that $F \leq_p H$. Indeed this is the case:

**Solved Exercise 13.2 — Transitivity of polynomial-time reductions.** For every $F, G, H : \{0,1\}^* \rightarrow \{0,1\}$, if $F \leq_p G$ and $G \leq_p H$ then $F \leq_p H$.

■

**Solution:**

If $F \leq_p G$ and $G \leq_p H$ then there exist polynomial-time computable functions $R_1$ and $R_2$ mapping $\{0,1\}^*$ to $\{0,1\}^*$ such that for every $x \in \{0,1\}^*$, $F(x) = G(R_1(x))$ and for every $y \in \{0,1\}^*$, $G(y) = H(R_2(y))$. Combining these two equalities, we see that for every $x \in \{0,1\}^*$, $F(x) = H(R_2(R_1(x)))$ and so to show that $F \leq_p H$, it is sufficient to show that the map $x \mapsto R_2(R_1(x))$ is computable in polynomial time. But if there are some constants $c, d$ such that $R_1(x)$ is computable in time $|x|^c$ and $R_2(y)$ is computable in time $|y|^d$ then $R_2(R_1(x))$ is computable in time $(|x|^c)^d = |x|^{cd}$ which is polynomial.

■

## 13.3 REDUCING 3SAT TO ZERO ONE EQUATIONS

We will now show our first example of a reduction. The *Zero-One Linear Equations problem* corresponds to the function $01EQ : \{0,1\}^* \rightarrow \{0,1\}$ whose input is a collection $E$ of linear equations in variables $x_0, \ldots, x_{n-1}$, and the output is 1 iff there is an assignment $x \in \{0,1\}^n$

of $0/1$ values to the variables that satisfies all the equations. For example, if the input $E$ is a string encoding the set of equations

$$\begin{aligned} x_0 + x_1 + x_2 &= 2 \\ x_0 + x_2 &= 1 \\ x_1 + x_2 &= 2 \end{aligned} \tag{13.3}$$

then $01EQ(E) = 1$ since the assignment $x = 011$ satisfies all three equations. We specifically restrict attention to linear equations in variables $x_0, \ldots, x_{n-1}$ in which every equation has the form $\sum_{i \in S} x_i = b$ where $S \subseteq [n]$ and $b \in \mathbb{N}$.[1]

[1] If you are familiar with matrix notation you may note that such equations can be written as $Ax = \mathbf{b}$ where $A$ is an $m \times n$ matrix with entries in $0/1$ and $\mathbf{b} \in \mathbb{N}^m$.

If we asked the question of whether there is a solution $x \in \mathbb{R}^n$ of *real numbers* to $E$, then this can be solved using the famous *Gaussian elimination* algorithm in polynomial time. However, there is no known efficient algorithm to solve $01EQ$. Indeed, such an algorithm would imply an algorithm for $3SAT$ as shown by the following theorem:

> **Theorem 13.2 — Hardness of $01EQ$.** $3SAT \leq_p 01EQ$

**Proof Idea:**

A constraint $x_2 \vee \overline{x}_5 \vee x_7$ can be written as $x_2 + (1 - x_5) + x_7 \geq 1$. This is a linear *inequality* but since the sum on the left-hand side is at most three, we can also turn it into an *equality* by adding two new variables $y, z$ and writing it as $x_2 + (1 - x_5) + x_7 + y + z = 3$. (We will use fresh such variables $y, z$ for every constraint.) Finally, for every variable $x_i$ we can add a variable $x'_i$ corresponding to its negation by adding the equation $x_i + x'_i = 1$, hence mapping the original constraint $x_2 \vee \overline{x}_5 \vee x_7$ to $x_2 + x'_5 + x_7 + y + z = 3$. The main **takeaway technique** from this reduction is the idea of adding *auxiliary variables* to replace an equation such as $x_1 + x_2 + x_3 \geq 1$ that is not quite in the form we want with the equivalent (for $0/1$ valued variables) equation $x_1 + x_2 + x_3 + u + v = 3$ which is in the form we want.

★

```python
def SAT2ZOE(φ):
    # Reduce 3SAT to 0/1 equations
    n = numvars(φ)
    E = ""
    for i in range(n): # add vars for negations
        E+= f"x{subscript(i)} + x{subscript(n+i)} = 1\n"
    m = 2*n
    for literals in getclauses(φ):
        # map each clause to equation
        def var(lit): # map literal to variable
            return f"x{subscript(n+int(lit[2:]))}" if lit[0]== "¬" else f"x{subscript(lit[1:])}"
        E+= " + ".join([var(lit) for lit in literals])
        E+= f" + x{subscript(m)} + x{subscript(m+1)} = 3\n"
        m += 2
    return Equation(E)
```

```python
SAT2ZOE("(x0 ∨ ¬x3 ∨ x2 ) ∧ (x0 ∨ x1 ∨ ¬x2 ) ∧ (x1 ∨ x2 ∨ ¬x3 )")
```

$$\begin{aligned} x_0 + x_4 &= 1 \\ x_1 + x_5 &= 1 \\ x_2 + x_6 &= 1 \\ x_3 + x_7 &= 1 \\ x_0 + x_7 + x_2 + x_8 + x_9 &= 3 \\ x_0 + x_1 + x_6 + x_{10} + x_{11} &= 3 \\ x_1 + x_2 + x_7 + x_{12} + x_{13} &= 3 \end{aligned}$$

**Figure 13.3**: Left: Python code implementing the reduction of $3SAT$ to $01EQ$. Right: Example output of the reduction. Code is in our repository.

*Proof of Theorem 13.2.* To prove the theorem we need to:

1. Describe an algorithm $R$ for mapping an input $\varphi$ for $3SAT$ into an input $E$ for $01EQ$.

2. Prove that the algorithm runs in polynomial time.

3. Prove that $01EQ(R(\varphi)) = 3SAT(\varphi)$ for every 3CNF formula $\varphi$.

We now proceed to do just that. Since this is our first reduction, we will spell out this proof in detail. However it straightforwardly follows the proof idea.

**Algorithm 13.3 — $3SAT$ to $01EQ$ reduction.**

**Input:** 3CNF formula $\varphi$ with $n$ variables $x_0, \ldots, x_{n-1}$ and $m$ clauses.

**Output:** Set $E$ of linear equations over $0/1$ such that $3SAT(\varphi) = 1$ -iff $01EQ(E) = 1$.

Let $E$'s variables be $x_0, \ldots, x_{n-1}$, $x'_0, \ldots, x'_{n-1}$, $y_0, \ldots, y_{m-1}$, $z_0, \ldots, z_{m-1}$.
**for** $i \in [n]$ **do**
  add to $E$ the equation $x_i + x'_i = 1$
**end for**
**for** j∈[m] **do**
  Let $j$-th clause be $w_1 \vee w_2 \vee w_3$ where $w_1, w_2, w_3$ are literals.
  **for** $a \in [3]$ **do**
    **if** $w_a$ is variable $x_i$ **then**
      set $t_a \leftarrow x_i$
    **end if**
    **if** $w_a$ is negation $\neg x_i$ **then**
      set $t_a \leftarrow x'_i$
    **end if**
  **end for**
  Add to $E$ the equation $t_1 + t_2 + t_3 + y_j + z_j = 3$.
**end for**
**return** $E$

The reduction is described in Algorithm 13.3, see also Fig. 13.3. If the input formula has $n$ variable and $m$ steps, Algorithm 13.3 creates a set $E$ of $n+m$ equations over $2n+2m$ variables. Algorithm 13.3 makes an initial loop of $n$ steps (each taking constant time) and then another loop of $m$ steps (each taking constant time) to create the equations, and hence it runs in polynomial time.

Let $R$ be the function computed by Algorithm 13.3. The heart of the proof is to show that for every 3CNF $\varphi$, $01EQ(R(\varphi)) = 3SAT(\varphi)$. We split the proof into two parts. The first part, traditionally known as the **completeness** property, is to show that if $3SAT(\varphi) = 1$ then $O1EQ(R(\varphi)) = 1$. The second part, traditionally known as the **soundness** property, is to show that if $01EQ(R(\varphi)) = 1$ then $3SAT(\varphi) = 1$.

(The names "completeness" and "soundness" derive viewing a solution to $R(\varphi)$ as a "proof" that $\varphi$ is satisfiable, in which case these conditions corresponds to completeness and soundness as defined in Section 10.1.1. However, if you find the names confusing you can simply think of completeness as the "1-instance maps to 1-instance" property and soundness as the "0-instance maps to 0-instance" property.)

We complete the proof by showing both parts:

- **Completeness:** Suppose that $3SAT(\varphi) = 1$, which means that there is an assignment $x \in \{0,1\}^n$ that satisfies $\varphi$. We know that for every clause $C_j$ in $\varphi$ of the form $w_1 \vee w_2 \vee w_3$ (with $w_1, w_2, w_3$ being literals), $w_1 + w_2 + w_3 \geq 1$, which means that we can assign values to $y_j, z_j$ in $\{0,1\}$ such that $w_1 + w_2 + w_3 + y_j + z_j = 3$. This means that if we let $x'_i = 1 - x_i$ for every $i \in [n]$, then the assignment $x_0, \ldots, x_{n-1}$, $x'_0, \ldots, x'_{n-1}, y_0, \ldots, y_{m-1}, z_0, \ldots, z_{m-1}$ satisfies the equations $E = R(\varphi)$ and hence $01EQ(R(\varphi)) = 1$.

- **Soundness:** Suppose that the set of equations $E = R(\varphi)$ has a satisfying assignment $x_0, \ldots, x_{n-1}, x'_0, \ldots, x'_{n-1}, y_0, \ldots, y_{m-1}, z_0, \ldots, z_{m-1}$. Then it must be the case that $x'_i$ is the negation of $x_i$ for all $i \in [n]$ and since $y_j + z_j \leq 2$ for every $j \in [m]$, it must be the case that for every clause $C_j$ in $\varphi$ of the form $w_1 \vee w_2 \vee w_3$ (with $w_1, w_2, w_3$ being literals), $w_1 + w_2 + w_3 \geq 1$, which means that the assignment $x_0, \ldots, x_{n-1}$ satisfies $\varphi$ and hence $3SAT(\varphi) = 1$.

■

**Anatomy of a reduction.** A reduction is simply an algorithm, and like any algorithm, when we come up with a reduction, it is not enough to describe *what* the reduction does, but we also have to provide an *analysis* of *why* it actually works. Specifically, to describe a reduction $R$ demonstrating that $F \leq_p G$ we need to provide the following:

- **Algorithm description:** This is the description of *how* the algorithm maps an input into the output. For example, Algorithm 13.3 above is the description of how we map an instance of $3SAT$ into an instance of $01EQ$ in the reduction demonstrating $3SAT \leq_p 01EQ$.

- **Algorithm analysis:** It is not enough to describe *how* the algorithm works but we need to also explain *why* it works. In particular we need to provide an *analysis* explaining why the reduction is both *efficient* (i.e., runs in polynomial time) and *correct* (satisfies that $G(R(x) = F(x)$ for every $x$)). Specifically, the components of analysis of a reduction $R$ include:

- **Efficiency:** We need to show that $R$ runs in polynomial time. In most reductions we encounter this part is straightforward, as the reductions we typically use involve a constant number of nested loops, each involving a constant number of operations.

- **Completeness:** In a reduction $R$ demonstrating $F \leq_p G$, the *completeness* condition is the condition that for every $x \in \{0,1\}^*$, if $F(x) = 1$ then $G(R(x)) = 1$. Typically we construct the reduction to ensure that this holds, by giving a way to map a "certificate/-solution" certifying that $F(x) = 1$ into a solution certifying that $G(R(x)) = 1$. For example in the proof of Theorem 13.2 the satisfying assignment for the $3SAT$ formula $\varphi$ can be mapped to a solution to the set of equations $R(\varphi)$.

- **Soundness:** This is the condition that if $F(x) = 0$ then $G(R(x)) = 0$ or (taking the contrapositive) if $G(R(x)) = 1$ then $F(x) = 1$. This is sometimes straightforward but can also be harder to show than the completeness condition, and in more advanced reductions (such as the reduction $SAT \leq_p ISET$ of Theorem 13.5) demonstrating soundness is the main part of the analysis.

Whenever you need to provide a reduction, you should make sure that your description has all these components. While it is sometimes tempting to weave together the description of the reduction and its analysis, it is usually clearer if you separate the two, and also break down the analysis to its three components of efficiency, completeness, and soundness.

### 13.3.1 Quadratic equations

Now that we reduced $3SAT$ to $01EQ$, we can use this to reduce $3SAT$ to the *quadratic equations* problem. This is the function $QUADEQ$ in which the input is a list of $n$-variate polynomials $p_0, \ldots, p_{m-1} : \mathbb{R}^n \to \mathbb{R}$ that are all of degree at most two (i.e., they are *quadratic*) and with integer coefficients. (The latter condition is for convenience and can be achieved by scaling.) We define $QUADEQ(p_0, \ldots, p_{m-1})$ to equal 1 if and only if there is a solution $x \in \mathbb{R}^n$ to the equations $p_0(x) = 0$, $p_1(x) = 0$, …, $p_{m-1}(x) = 0$.

For example, the following is a set of quadratic equations over the variables $x_0, x_1, x_2$:

$$
\begin{aligned}
x_0^2 - x_0 &= 0 \\
x_1^2 - x_1 &= 0 \\
x_2^2 - x_2 &= 0 \\
1 - x_0 - x_1 + x_0 x_1 &= 0
\end{aligned}
\tag{13.4}
$$

You can verify that $x \in \mathbb{R}^3$ satisfies this set of equations if and only if $x \in \{0,1\}^3$ and $x_0 \vee x_1 = 1$.

**Theorem 13.4 — Hardness of quadratic equations.**

$$3SAT \leq_p QUADEQ \tag{13.5}$$

**Proof Idea:**

Using the transitivity of reductions (Solved Exercise 13.2), it is enough to show that $01EQ \leq_p QUADEQ$, but this follows since we can phrase the equation $x_i \in \{0,1\}$ as the quadratic constraint $x_i^2 - x_i = 0$. The **takeaway technique** of this reduction is that we can use *nonlinearity* to force continuous variables (e.g., variables taking values in $\mathbb{R}$) to be discrete (e.g., take values in $\{0,1\}$).

⋆

*Proof of Theorem 13.4.* By Theorem 13.2 and Solved Exercise 13.2, it is sufficient to prove that $01EQ \leq_p QUADEQ$. Let $E$ be an instance of $01EQ$ with variables $x_0, \ldots, x_{m-1}$. We define $R(E)$ to be the set of quadratic equations $E'$ that is obtained by taking the linear equations in $E$ and adding to them the $n$ quadratic equations $x_i^2 - x_i = 0$ for all $i \in [n]$. Clearly the map $E \mapsto E'$ can be computed in polynomial time. We claim that $01EQ(E) = 1$ if and only if $QUADEQ(E') = 1$. Indeed, the only difference between the two instances is that:

- In the $01EQ$ instance $E$, the equations are over variables $x_0, \ldots, x_{n-1}$ in $\{0,1\}$.
- In the $QUADEQ$ instance $E'$, the equations are overvariables $x_0, \ldots, x_{n-1} \in \mathbb{R}$ but we have the extra constraints $x_i^2 - x_i = 0$ for all $i \in [n]$.

Since for every $a \in \mathbb{R}$, $a^2 - a = 0$ if and only if $a \in \{0,1\}$, the two sets of equations are equivalent and $01EQ(E) = QUADEQ(E')$ which is what we wanted to prove.

■

## 13.4 THE INDEPENDENT SET PROBLEM

For a graph $G = (V, E)$, an independent set (also known as a *stable set*) is a subset $S \subseteq V$ such that there are no edges with both endpoints in $S$ (in other words, $E(S,S) = \emptyset$). Every "singleton" (set consisting of a single vertex) is trivially an independent set, but finding larger independent sets can be challenging. The *maximum independent set* problem (henceforth simply "independent set") is the task of finding the largest independent set in the graph. The independent set problem is naturally related to *scheduling problems*: if we put an edge between two conflicting tasks, then an independent set corresponds to a set of tasks that can all be scheduled together without conflicts. The independent set problem has been studied in a variety of settings, including for example in the case of algorithms for finding structure in protein-protein interaction graphs.

As mentioned in Section 13.1, we think of the independent set problem as the function $ISET : \{0,1\}^* \rightarrow \{0,1\}$ that on input a graph $G$ and a number $k$ outputs 1 if and only if the graph $G$ contains an independent set of size at least $k$. We now reduce 3SAT to Independent set.

> **Theorem 13.5 — Hardness of Independent Set.** $3SAT \leq_p ISET$.

**Proof Idea:**

The idea is that finding a satisfying assignment to a 3SAT formula corresponds to satisfying many local constraints without creating any conflicts. One can think of "$x_{17} = 0$" and "$x_{17} = 1$" as two conflicting events, and of the constraints $x_{17} \vee \overline{x}_5 \vee x_9$ as creating a conflict between the events "$x_{17} = 0$", "$x_5 = 1$" and "$x_9 = 0$", saying that these three cannot simultaneosly co-occur. Using these ideas, we can we can think of solving a 3SAT problem as trying to schedule non conflicting events, though the devil is, as usual, in the details. The **takeaway technique** here is to map each clause of the original formula into a *gadget* which is a small subgraph (or more generally "subinstance") satisfying some convenient properties. We will see these "gadgets" used time and again in the construction of polynomial-time reductions.

⋆

*Proof of Theorem 13.5.* Given a 3SAT formula $\varphi$ on $n$ variables and with $m$ clauses, we will create a graph $G$ with $3m$ vertices as follows. (See Fig. 13.4 for an example and Fig. 13.5 for Python code.)

- A clause $C$ in $\varphi$ has the form $C = y \vee y' \vee y''$ where $y, y', y''$ are *literals* (variables or their negation). For each such clause $C$, we will add three vertices to $G$, and label them $(C, y)$, $(C, y')$, and $(C, y'')$ respectively. We will also add the three edges between all pairs of these vertices, so they form a *triangle*. Since there are $m$ clauses in $\varphi$, the graph $G$ will have $3m$ vertices.

- In addition to the above edges, we also add an edge between every pair vertices of the form $(C, y)$ and $(C', y')$ where $y$ and $y'$ are *conflicting* literals. That is, we add an edge between $(C, y)$ and $(C, y')$ if there is an $i$ such that $y = x_i$ and $y' = \overline{x}_i$ or vice versa.

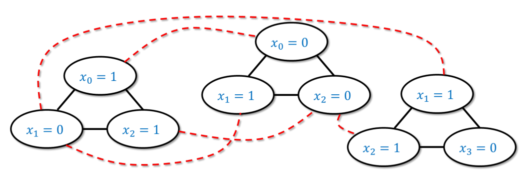



**Figure 13.4**: An example of the reduction of $3SAT$ to $ISET$ for the case the original input formula is $\varphi = (x_0 \vee \overline{x}_1 \vee x_2) \wedge (\overline{x}_0 \vee x_1 \vee \overline{x}_2) \wedge (x_1 \vee x_2 \vee \overline{x}_3)$. We map each clause of $\varphi$ to a triangle of three vertices, each tagged above with "$x_i = 0$" or "$x_i = 1$" depending on the value of $x_i$ that would satisfy the particular literal. We put an edge between every two literals that are *conflicting* (i.e., tagged with "$x_i = 0$" and "$x_i = 1$" respectively).

The above construction of $G$ based on $\varphi$ can clearly be carried out in polynomial time. Hence to prove the theorem we need to show that $\varphi$ is satisfiable if and only if $G$ contains an independent set of $m$ vertices. We now show both directions of this equivalence:

**Part 1: Completeness.** The "completeness" direction is to show that if $\varphi$ has a satisfying assignment $x^*$, then $G$ has an independent set $S^*$ of $m$ vertices. Let us now show this.

Indeed, suppose that $\varphi$ has a satisfying assignment $x^* \in \{0,1\}^n$. Then for every clause $C = y \vee y' \vee y''$ of $\varphi$, one of the literals $y, y', y''$ must evaluate to *true* under the assignment $x^*$ (as otherwise it would not satisfy $\varphi$). We let $S$ be a set of $m$ vertices that is obtained by choosing for every clause $C$ one vertex of the form $(C, y)$ such that $y$ evaluates to true under $x^*$. (If there is more than one such vertex for the same $C$, we arbitrarily choose one of them.)

We claim that $S$ is an independent set. Indeed, suppose otherwise that there was a pair of vertices $(C, y)$ and $(C', y')$ in $S$ that have an edge between them. Since we picked one vertex out of each triangle corresponding to a clause, it must be that $C \neq C'$. Hence the only way that there is an edge between $(C, y)$ and $(C, y')$ is if $y$ and $y'$ are conflicting literals (i.e. $y = x_i$ and $y' = \overline{x}_i$ for some $i$). But that would that they can't both evaluate to *true* under the assignment $x^*$, which contradicts the way we constructed the set $S$. This completes the proof of the completeness condition.

**Part 2: Soundness.** The "soundness" direction is to show that if $G$ has an independent set $S^*$ of $m$ vertices, then $\varphi$ has a satisfying assignment $x^* \in \{0,1\}^n$. Let us now show this.

Indeed, suppose that $G$ has an independent set $S*$ with $m$ vertices. We will define an assignment $x^* \in \{0,1\}^n$ for the variables of $\varphi$ as follows. For every $i \in [n]$, we set $x^*_i$ according to the following rules:

- If $S^*$ contains a vertex of the form $(C, x_i)$ then we set $x^*_i = 1$.
- If $S^*$ contains a vertex of the form $(C, \overline{x_i})$ then we set $x^*_i = 0$.
- If $S^*$ does not contain a vertex of either of these forms, then it does not matter which value we give to $x^*_i$, but for concreteness we'll set $x^*_i = 0$.

The first observation is that $x^*$ is indeed well defined, in the sense that the rules above do not conflict with one another, and ask to set $x^*_i$ to be both $0$ and $1$. This follows from the fact that $S^*$ is an *independent set* and hence if it contains a vertex of the form $(C, x_i)$ then it cannot contain a vertex of the form $(C', \overline{x_i})$.

We now claim that $x^*$ is a satisfying assignment for $\varphi$. Indeed, since $S^*$ is an independent set, it cannot have more than one vertex inside each one of the $m$ triangles $(C, y), (C, y'), (C, y'')$ corresponding to a clause of $\varphi$. Hence since $|S^*| = m$, it must have exactly one vertex in each such triangle. For every clause $C$ of $\varphi$, if $(C, y)$ is the vertex in $S^*$ in the triangle corresponding to $C$, then by the way we defined $x^*$, the literal $y$ must evaluate to *true*, which means that $x^*$ satisfies this clause. Therefore $x^*$ satisfies all clauses of $\varphi$, which is the definition of a satisfying assignment.

This completes the proof of Theorem 13.5

■


**Figure 13.5**: The reduction of 3SAT to Independent Set. On the righthand side is *Python* code that implements this reduction. On the lefthand side is a sample output of the reduction. We use black for the "triangle edges" and red for the "conflict edges". Note that the satisfying assignment $x^* = 0110$ corresponds to the independent set $(0, \neg x_3)$, $(1, \neg x_0)$, $(2, x_2)$.

**Solved Exercise 13.3 — Clique is equivalent to independent set.** The maximum clique problem corresponds to the function $CLIQUE : \{0,1\}^* \rightarrow \{0,1\}$ such that for a graph $G$ and a number $k$, $CLIQUE(G, k) = 1$ iff there is a $S$ subset of $k$ vertices such that for *every* distinct $u, v \in S$, the edge $u, v$ is in $G$. Such a set is known as a *clique*.

Prove that $CLIQUE \leq_p ISET$ and $ISET \leq_p CLIQUE$.

■

**Solution:**

If $G = (V, E)$ is a graph, we denote by $\overline{G}$ its *complement* which is the graph on the same vertices $V$ and such that for every distinct $u, v \in V$, the edge $\{u, v\}$ is present in $\overline{G}$ if and only if this edge is *not* present in $G$.

This means that for every set $S$, $S$ is an independent set in $G$ if and only if $S$ is a *clique* in $\overline{S}$. Therefore for every $k$, $ISET(G, k) = CLIQUE(\overline{G}, k)$. Since the map $G \mapsto \overline{G}$ can be computed efficiently, this yields a reduction $ISET \leq_p CLIQUE$. Moreover, since $\overline{\overline{G}} = G$ this yields a reduction in the other direction as well.

■

## 13.5 REDUCING INDEPENDENT SET TO MAXIMUM CUT

We now show that the independent set problem reduces to the *maximum cut* (or "max cut") problem, modeled as the function *MAXCUT* that on input a pair $(G, k)$ outputs 1 iff $G$ contains a cut of at least $k$ edges. Since both are graph problems, a reduction from independent set to max cut maps one graph into the other, but as we will see the output graph does not have to have the same vertices or edges as the input graph.

**Theorem 13.6 — Hardness of Max Cut.** $ISET \leq_p MAXCUT$

**Proof Idea:**

We will map a graph $G$ into a graph $H$ such that a large independent set in $G$ becomes a partition cutting many edges in $H$. We can think of a cut in $H$ as coloring each vertex either "blue" or "red". We will add a special "source" vertex $s^*$, connect it to all other vertices, and assume without loss of generality that it is colored blue. Hence the more vertices we color red, the more edges from $s^*$ we cut. Now, for every edge $u, v$ in the original graph $G$ we will add a special "gadget" which will be a small subgraph that involves $u$,$v$, the source $s^*$, and two other additional vertices. We design the gadget in a way so that if the red vertices are not an independent set in $G$ then the corresponding cut in $H$ will be "penalized" in the sense that it would not cut as many edges. Once we set for ourselves this objective, it is not hard to find a gadget that achieves it— see the proof below. Once again the **takeaway technique** is to use (this time a slightly more clever) gadget.

⋆

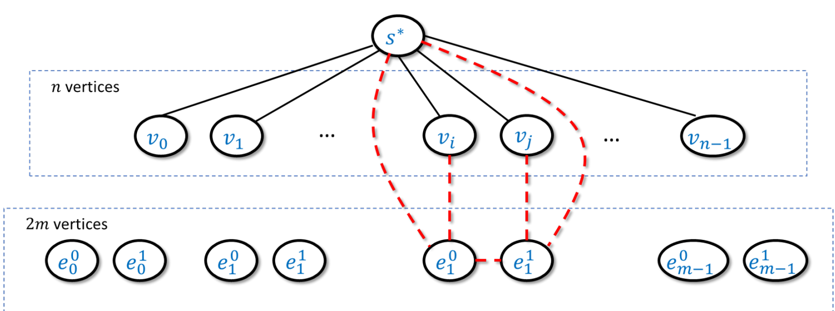



**Figure 13.6**: In the reduction of $ISET$ to $MAXCUT$ we map an $n$-vertex $m$-edge graph $G$ into the $n + 2m + 1$ vertex and $n + 5m$ edge graph $H$ as follows. The graph $H$ contains a special "source" vertex $s^*$,$n$ vertices $v_0, \ldots, v_{n-1}$, and $2m$ vertices $e_0^0, e_0^1, \ldots, e_{m-1}^0, e_{m-1}^1$ with each pair corresponding to an edge of $G$. We put an edge between $s^*$ and $v_i$ for every $i \in [n]$, and if the $t$-th edge of $G$ was $(v_i, v_j)$ then we add the five edges $(s^*, e_t^0), (s^*, e_t^1), (v_i, e_t^0), (v_j, e_t^1), (e_t^0, e_t^1)$. The intent is that if cut at most one of $v_i, v_j$ from $s^*$ then we'll be able to cut 4 out of these five edges, while if we cut both $v_i$ and $v_j$ from $s^*$ then we'll be able to cut at most three of them.

*Proof of Theorem 13.6.* We will transform a graph $G$ of $n$ vertices and $m$ edges into a graph $H$ of $n + 1 + 2m$ vertices and $n + 5m$ edges in the following way (see also Fig. 13.6). The graph $H$ contains all vertices of $G$ (though not the edges between them!) and in addition $H$ also has:

* A special vertex $s^*$ that is connected to all the vertices of $G$

* For every edge $e = \{u, v\} \in E(G)$, two vertices $e_0, e_1$ such that $e_0$ is connected to $u$ and $e_1$ is connected to $v$, and moreover we add the edges $\{e_0, e_1\}, \{e_0, s^*\}, \{e_1, s^*\}$ to $H$.

Theorem 13.6 will follow by showing that $G$ contains an independent set of size at least $k$ if and only if $H$ has a cut cutting at least $k + 4m$ edges. We now prove both directions of this equivalence:

**Part 1: Completeness.** If $I$ is an independent $k$-sized set in $G$, then we can define $S$ to be a cut in $H$ of the following form: we let $S$ contain all the vertices of $I$ and for every edge $e = \{u, v\} \in E(G)$, if $u \in I$ and $v \notin I$ then we add $e_1$ to $S$; if $u \notin I$ and $v \in I$ then we add $e_0$ to $S$; and if $u \notin I$ and $v \notin I$ then we add both $e_0$ and $e_1$ to $S$. (We don't need to worry about the case that both $u$ and $v$ are in $I$ since it is an independent set.) We can verify that in all cases the number of edges from $S$ to its complement in the gadget corresponding to $e$ will be four (see Fig. 13.7). Since $s^*$ is not in $S$, we also have $k$ edges from $s^*$ to $I$, for a total of $k + 4m$ edges.

**Part 2: Soundness.** Suppose that $S$ is a cut in $H$ that cuts at least $C = k + 4m$ edges. We can assume that $s^*$ is not in $S$ (otherwise we can "flip" $S$ to its complement $\overline{S}$, since this does not change the size of the cut). Now let $I$ be the set of vertices in $S$ that correspond to the original vertices of $G$. If $I$ was an independent set of size $k$ then would be done. This might not always be the case but we will see that if $I$ is not an independent set then it's also larger than $k$. Specifically, we define $m_{in} = |E(I, I)|$ be the set of edges in $G$ that are contained in $I$ and let $m_{out} = m - m_{in}$ (i.e., if $I$ is an independent set then $m_{in} = 0$ and $m_{out} = m$). By the properties of our gadget we know that for every edge $\{u, v\}$ of $G$, we can cut at most three edges when both $u$ and $v$ are in $S$, and at most four edges otherwise. Hence the number $C$ of edges cut by $S$ satisfies $C \leq |I| + 3m_{in} + 4m_{out} = |I| + 3m_{in} + 4(m - m_{in}) = |I| + 4m - m_{in}$. Since $C = k + 4m$ we get that $|I| - m_{in} \geq k$. Now we can transform $I$ into an independent set $I'$ by going over every one of the $m_{in}$ edges that are inside $I$ and removing one of the endpoints of the edge from it. The resulting set $I'$ is an independent set in the graph $G$ of size $|I| - m_{in} \geq k$ and so this concludes the proof of the soundness condition.

■

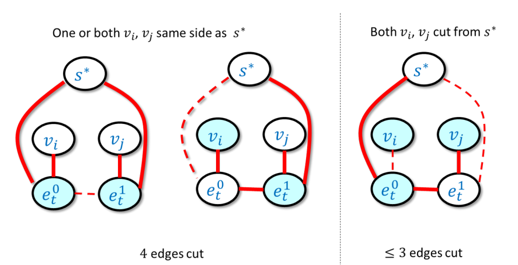
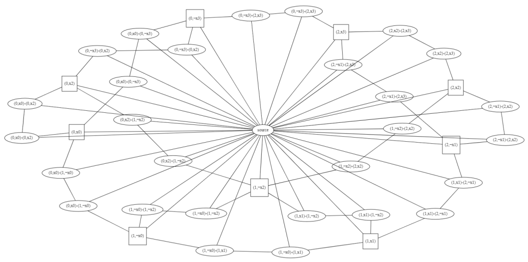



**Figure 13.7**: In the reduction of independent set to max cut, for every $t \in [m]$, we have a "gadget" corresponding to the $t$-th edge $e = \{v_i, v_j\}$ in the original graph. If we think of the side of the cut containing the special source vertex $s^*$ as "white" and the other side as "blue", then the leftmost and center figures show that if $v_i$ and $v_j$ are not both blue then we can cut four edges from the gadget. In contrast, by enumerating all possibilities one can verify that if both $u$ and $v$ are blue, then no matter how we color the intermediate vertices $e_t^0, e_t^1$, we will cut at most three edges from the gadget.

## 13.6 REDUCING 3SAT TO LONGEST PATH

**Note:** This section is still a little messy; feel free to skip it or just read it without going into the proof details. The proof appears in Section 7.5 in Sipser's book.

One of the most basic algorithms in Computer Science is Dijkstra's algorithm to find the *shortest path* between two vertices. We now show



```
# Reduction IS to MAXCUT
def IS2MAXCUT(G):
    G =nxgraph(G)
    H =Graph(engine='sfdp')
    s ="source"
    H.node(s) # create source node

    for v in G.nodes():
        H.node(v,shape="square")
        H.edge(s,v)

    j =0
    for (u,v) in G.edges():
        g1 = "e"+str(j)+"a"
        g2 = "e"+str(j)+"b"
        H.node(g1,label=u+"-"+v)
        H.node(g2,label=u+"-"+v)
        # add gadget
        H.edges([(s,g1),(s,g2),(g1,g2),(u,g1),(v,g2)])
        j +=1


    return H
```

**Figure 13.8**: The reduction of independent set to max cut. On the righthand side is Python code implementing the reduction. On the lefthand side is an example output of the reduction where we apply it to the independent set instance that is obtained by running the reduction of Theorem 13.5 on the 3CNF formula $(x_0 \vee \overline{x}_3 \vee x_2) \wedge (\overline{x}_0 \vee x_1 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2 \vee x_3)$.

that in contrast, an efficient algorithm for the *longest path* problem would imply a polynomial-time algorithm for 3SAT.

**Theorem 13.7 — Hardness of longest path.**

$$3SAT \leq_p LONGPATH \tag{13.6}$$

**Proof Idea:**

To prove Theorem 13.7 need to show how to transform a 3CNF formula $\varphi$ into a graph $G$ and two vertices $s, t$ such that $G$ has a path of length at least $k$ if and only if $\varphi$ is satisfiable. The idea of the reduction is sketched in Fig. 13.9 and Fig. 13.10. We will construct a graph that contains a potentially long "snaking path" that corresponds to all variables in the formula. We will add a "gadget" corresponding to each clause of $\varphi$ in a way that we would only be able to use the gadgets if we have a satisfying assignment.

⋆

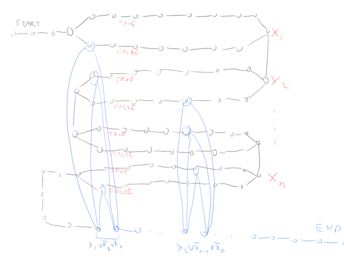


**Figure 13.9**: We can transform a 3SAT formula $\varphi$ into a graph $G$ such that the longest path in the graph $G$ would correspond to a satisfying assignment in $\varphi$. In this graph, the black colored part corresponds to the variables of $\varphi$ and the blue colored part corresponds to the vertices. A sufficiently long path would have to first "snake" through the black part, for each variable choosing either the "upper path" (corresponding to assigning it the value `True`) or the "lower path" (corresponding to assigning it the value `False`). Then to achieve maximum length the path would traverse through the blue part, where to go between two vertices corresponding to a clause such as $x_{17} \vee \overline{x}_{32} \vee x_{57}$, the corresponding vertices would have to have been not traversed before.

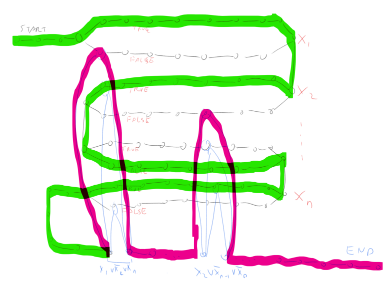


**Figure 13.10**: The graph above with the longest path marked on it, the part of the path corresponding to variables is in green and part corresponding to the clauses is in pink.

*Proof of Theorem 13.7.* We build a graph $G$ that "snakes" from $s$ to $t$ as follows. After $s$ we add a sequence of $n$ long loops. Each loop has an "upper path" and a "lower path". A simple path cannot take both the upper path and the lower path, and so it will need to take exactly one of them to reach $s$ from $t$.

Our intention is that a path in the graph will correspond to an assignment $x \in \{0,1\}^n$ in the sense that taking the upper path in the $i^{th}$ loop corresponds to assigning $x_i = 1$ and taking the lower path corresponds to assigning $x_i = 0$. When we are done snaking through all the $n$ loops corresponding to the variables to reach $t$ we need to pass through $m$ "obstacles": for each clause $j$ we will have a small gadget consisting of a pair of vertices $s_j, t_j$ that have three paths between them. For example, if the $j^{th}$ clause had the form $x_{17} \vee \overline{x}_{55} \vee x_{72}$ then one path would go through a vertex in the lower loop corresponding to $x_{17}$, one path would go through a vertex in the upper loop corresponding to $x_{55}$ and the third would go through the lower loop corresponding to $x_{72}$. We see that if we went in the first stage according to a satisfying assignment then we will be able to find a free vertex to travel from $s_j$ to $t_j$. We link $t_1$ to $s_2$, $t_2$ to $s_3$, etc and link $t_m$ to $t$. Thus a satisfying assignment would correspond to a path from $s$ to $t$ that goes through one path in each loop corresponding to the variables, and one path in each loop corresponding to the clauses. We can make the loop corresponding to the variables long enough so that we must take the entire path in each loop in order to have a fighting chance of getting a path as long as the one corresponds to a satisfying assignment. But if we do that, then the only way if we are able to reach $t$ is if the paths we took corresponded to a satisfying assignment, since otherwise we will have one clause $j$ where we cannot reach $t_j$ from $s_j$ without using a vertex we already used before.

■

### 13.6.1 Summary of relations

We have shown that there are a number of functions $F$ for which we can prove a statement of the form "If $F \in \mathbf{P}$ then $3SAT \in \mathbf{P}$". Hence coming up with a polynomial-time algorithm for even one of these problems will entail a polynomial-time algorithm for $3SAT$ (see for example Fig. 13.11). In Chapter 14 we will show the inverse direction ("If $3SAT \in \mathbf{P}$ then $F \in \mathbf{P}$") for these functions, hence allowing us to conclude that they have *equivalent complexity* to $3SAT$.

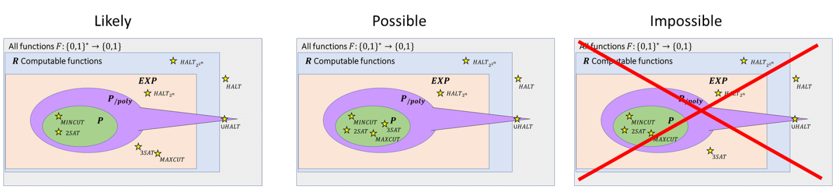



**Figure 13.11**: So far we have shown that $\mathbf{P} \subseteq \mathbf{EXP}$ and that several problems we care about such as $3SAT$ and $MAXCUT$ are in $\mathbf{EXP}$ but it is not known whether or not they are in $\mathbf{EXP}$. However, since $3SAT \leq_p MAXCUT$ we can rule out the possiblity that $MAXCUT \in \mathbf{P}$ but $3SAT \notin \mathbf{P}$. The relation of $\mathbf{P}_{/\text{poly}}$ to the class $\mathbf{EXP}$ is not known. We know that $\mathbf{EXP}$ does not contain $\mathbf{P}_{/\text{poly}}$ since the latter even contains uncomputable functions, but we do not know whether ot not $\mathbf{EXP} \subseteq \mathbf{P}_{/\text{poly}}$ (though it is believed that this is not the case and in particular that both $3SAT$ and $MAXCUT$ are not in $\mathbf{P}_{/\text{poly}}$).

✓ **Lecture Recap**

- The computational complexity of many seemingly unrelated computational problems can be related to one another through the use of *reductions*.
- If $F \leq_p G$ then a polynomial-time algorithm for $G$ can be transformed into a polynomial-time algorithm for $F$.
- Equivalently, if $F \leq_p G$ and $F$ does *not* have a polynomial-time algorithm then neither does $G$.
- We've developed many techniques to show that $3SAT \leq_p F$ for interesting functions $F$. Sometimes we can do so by using *transitivity* of reductions: if $3SAT \leq_p G$ and $G \leq_p F$ then $3SAT \leq_p F$.

## 13.7 EXERCISES

## 13.8 BIBLIOGRAPHICAL NOTES

Several notions of reductions are defined in the literature. The notion defined in Definition 13.1 is often known as a *mapping reduction*, *many to one reduction* or a *Karp reduction*.

The *maximal* (as opposed to *maximum*) independent set is the task of finding a "local maximum" of an independent set: an independent set $S$ such that one cannot add a vertex to it without losing the independence property (such a set is known as a *vertex cover*). Unlike finding a *maximum* independent set, finding a *maximal* independent set can be done efficiently by a greedy algorithm, but this local maximum can be much smaller than the global maximum.

Reduction of independent set to max cut taken from these notes. Image of Hamiltonian Path through Dodecahedron by Christoph Sommer.

We have mentioned that the line between reductions used for algorithm design and showing hardness is sometimes blurry. An excellent example for this is the area of *SAT Solvers* (see [Gom+08]). In this field people use algorithms for SAT (that take exponential time in the worst case but often are much faster on many instances in practice) together with reductions of the form $F \leq_p SAT$ to derive algorithms for other functions $F$ of interest.