Week 1: (Un)Natural Numbers

The problems are designed to develop your skills with formal definitions and provide some practice with inductive reasoning.

> **Collaboration Policy:** You should work on the problems yourself, before discussing with others, and with your cohorts are your cohort meeting. By the Assessed Cohort Meeting, you and all of your cohortmates, should be prepared to present and discuss solutions to all of the assigned problems. In addition to discussing with your cohortmates, you may discuss the problems with anyone you want, and use any resources you want **except for any materials from previous offerings of this course**, which are not permitted.

Problem 1 *Cohortroductions*

Everyone in your cohort should introduce themselves, and teach all the other cohort members how to pronounce your name correctly, and share something interesting about yourself with the group. This can be anything you want, but a few suggestions:

– What did you do over the summer?

– What are the best and worst things about living under the Covid lockdown?

– What is the most interesting thing about your hometown?

– Who is your spirit animal?

– If you could change anything about UVA, what would you change?

As the assessed cohort meeting, this will be treated like other questions: *one* of you will be selected, and will be expected to introduce yourself and the other members of your cohort, including pronouncing their names correctly and recounting something interesting you learned about each other cohort member.

Problem 2 *Cohort Namesake*

Each cohort has been named after someone who is (at least loosely) connected to theoretical computer science. You are not expected to do an extensive research project here, but find out something about the people your cohort is named after.

At your cohort preparation meeting, everyone should share what they learned, and the group should settle a few interesting things that you'll share with your Cohort Leader at the Assessed Cohort Meeting.

As an additional step for this problem, you should also agree on one interesting fact about, and the best URL you found, about your cohort namesake, and one of your cohortees should post a message in `#week1` (changed from ~~`#general`~~) on discord to share this with the rest of the class.

Problem 3 *Higher Induction Practice*

Prove that any binary tree of height h has at most 2^{h-1} leaves.

Note: We haven't defined a *binary tree* (and the book doesn't). An adequate answer to this question will use the informal understanding of a binary tree which we expect you have entering this class (a tree where each node has 0, 1, or 2 children), but an excellent answer will include a definition of a binary tree and connect your proof to that definition.

Problem 4 *Addition is Commutative*

For this problem, we will use the successor definition of Natural Numbers from the *Constructing the Natural Numbers* video:

Definition 1 (Natural Numbers) We define the *Natural Numbers* as:

1. **0** is a Natural Number.
2. If n is a Natural Number, **S**(n) is a Natural Number.

We will use this definition of addition (from *Defining Addition*):

Definition 2 (Sum) The *sum* of two Natural Numbers a and b (denoted as $a + b$) is defined as:

1. If a is **0**, then $a + b$ is b.
2. Otherwise, a is **S**(p) for some Natural Number p, and $a + b$ is **S**$(p + b)$.

Prove that addition (as defined above) is *commutative* (that is, for all Natural Numbers a and b, $a + b$ is $b + a$). Note that what "is" means here is they are exactly the same representation (we are not using $=$, since we haven't defined it for our number representation). You can think of all the operations we have defined as just manipulating strings of symbols, and "x is y" meaning that x and y are exactly the same sequences of symbols.

Problem 5 *Countable Programs*

Prove that the set of all Python programs that you can execute on your laptop is *countable*.

Problem 6 *Powerset Proof*

The *Countable Sets* video set up a proof by induction that, for all *finite* sets S, $|\text{pow}(S)| = 2^{|S|}$, and we provided our proof. For this problem, you are expected to understand that proof well, and be able to answer questions about it during the assessed cohort meeting such as (but not limited to) "what is the predicate, P?", "explain step 1 of the Inductive Case?", "why is the base case the empty set?".

