Practice Problems Comments

Practice 1 *Relation Properties*

Considered the relation, $\leq$ (less than or equal to, with the standard meaning), with the domain set, $\mathbb{N}$ and codomain set $\mathbb{N}$. Which of these properties does the $\leq$ relation have: function, total, injective, surjective, bijective?

Solution:

Note we could consider $\leq$ as a function from an ordered pair of natural numbers to a Boolean, but here we consider it as a relation where there is an edge from each domain element to all the codomain elements it is less than or equal to (this wasn't clearly stated in the question, so there are lots of other ways you could have reasonably interpreted the $\leq$ relation.

So, there is an edge from $a \in \mathbb{N}$ to $b \in \mathbb{N}$ if and only if $a \leq b$. Since for a given $a \in \mathbb{N}$, there can be more than one $b \in \mathbb{N}$ such that $a \leq b$ this means there can be multiple edges out of a domain element, so it is not a function. It is total since there is at least one edge out of every element of $\mathbb{N}$ (note that $<$ would not be total, since 0 is not less than any element of the codomain).

Injective means ≤ 1 edge into every codomain element. It is not injective — for example, the codomain element 2 has incoming edges from three domain elements (0, 1, and 2). It is surjective (≥ 1 edge into every codomain element), since every element of the codomain has at least one domain element that is $\leq$ it. It is not bijective, since to be bijective it must be both injective and surjective, but it is not injective.

Practice 2 *Set Cardinality*

a. Assume $R : A \to B$ is an *total injective* function between A and B. What must be true about the relationship between $|A|$ and $|B|$?

   Solution: Since R is a total function, there is exactly one edge out of each domain element. To be injective (≤ 1 in), there cannot be multiple edges into any codomain elements (but there can be codomain elements with 0 incoming edges). So, every element of the domain is connected to a different element in the codomain. This means $|A| \leq |B|$.

b. Assume $R : A \to B$ is an *total surjective* function between A and B. What must be true about the relationship between $|A|$ and $|B|$?

   Solution: Surjective means ≥ 1 in, so $|A| \geq |B|$.

c. Assume $R : A \to B$ is a (not necessarily total) *surjective* function between A and B. What must be true about the relationship between $|A|$ and $|B|$?

   Solution: Still, $|A| \geq |B|$. It doesn't matter that R is not necessarily total, since if there are additional elements in A that have no out edges, that just means it is "more bigger". It does matter that R is a function — otherwise, all the out edges (to cover all of B) could be coming from a single domain element.

Practice 3 *Countable and Uncountable Infinities*

a. Prove that the integers, i.e., $\ldots, -2, -1, 0, 1, 2, \ldots$, are countably infinite.

   Solution: See Class 10.

b. Prove that the number of total injective functions between $\mathbb{N}$ and $\mathbb{N}$ is **uncountable**.

   Solution:

   As stated originally (with *countable* instead of uncountable), this one was impossible since the cardinality of the set is actually uncountable.

   A good problem solving strategy when a question seem hard to answer (and indeed, this question is quite difficult!) is to first start with an easier, but related question. (At least in this class, and it nearly all "real world" contexts, you are much better off answering a question by saying that you can't figure out how to solve the original question, but here's a variation on it that you can answer, than seeing a clearly bogus and deceptive answer to the original question.) So, first let's answer an easier variation of this question where instead of counting total, injective functions, we count all the binary relations.

   A binary relation is a subset of all possible edges between elements in the domain and codomain: $R \subset \mathbb{N} \times \mathbb{N}$. The cardinality of $\mathbb{N} \times \mathbb{N}$ is the same as the cardinality of $\mathbb{N}$, it is countably infinite. To see this, draw the product set in a grid and consider a bijection like, $0 \longleftrightarrow (0,0); 1 \longleftrightarrow (0,1); 2 \longleftrightarrow (1,0); 3 \longleftrightarrow (2,0); 4 \longleftrightarrow (1,1); 5 \longleftrightarrow (0,2); 6 \longleftrightarrow (0,3); 7 \longleftrightarrow (1,2); 8 \longleftrightarrow (2,1); 9 \longleftrightarrow (3,0); \cdots$.

   But, we can define a relation for any subset of the set $\mathbb{N} \times \mathbb{N}$. The set of all subsets of a set is its powerset, so we have the powerset of a countably infinite set which is uncountable. (Unlike in graphs, the labels matter here, since we are mapping between actual elements of the domain and codomain, so it is easy to see that all of these relations are different.)

   If you got this part, you are well prepared for the exam (which won't have any questions as hard as the one asked here on it). To answer the (corrected) original question, we need to think carefully about how to construct the total injective functions.

   Let's divide the codomain into two sets, both of which are countably infinite. An easy way to do this is to split it into even and odd numbers: $\mathbb{N} = EVENS \cup ODDS$ where $EVENS = \{0, 2, 4, \ldots\}$ and $ODDS = \{1, 3, 5, \ldots\}$. Since $EVENS$ is countably infinite, there is a total, injective function between $\mathbb{N}$ and $EVENS$.

   Now, we will show how to make a different total, injective function between $\mathbb{N}$ and each set in $\forall X \in pow(ODDS) \cup EVENS$. Since the cardinality of $pow(ODDS)$ is uncountable, if we can show a way to construct a different total, injective function for each element of X, we have showing that the cardinality of the set of all total, injective, functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

   I think the way I constructed the mapping in class was flawed, since I tried to use all the $EVENS$ before connecting the elements of $X \in pow(ODDS)$. This doesn't work since we've already used up all the domain elements in mapping to the $EVENS$. Instead, we need to interleave the two codomain subsets: alternate between elements of $EVENS$ and elements of X until X is used up (which might never happen since X can be infinite), mapping each to the next element in the domain.

   So, for example if $X = \{1, 3\}$, we would create the relation $0 \to 0, 1 \to 1, 2 \to 2, 3 \to 3, 4 \to 4, 5 \to 6, 6 \to 8, \ldots$. Thus, for each different element of $pow(ODDS)$ we construct a different binary relation which is still a total, injective function from $\mathbb{N}$ to $\mathbb{N}$, thus showing that the number of different total, injective functions is uncountable.

Note that this is not showing that its cardinality is the same as the cardinality of $pow(\mathbb{N})$ by itself (the question didn't ask for this, just to show that it is uncountable), but we can easily combine the proof we did for the simpler question to obtain this result. Since we showed that the cardinality of the set of all binary relations is $|pow(\mathbb{N})|$, and the set of total, injective functions is a subset of this set its cardinality must be $\leq$ the cardinality of its superset. Combined with the proof that it is uncountable, this shows that it has the same cardinality as $pow(\mathbb{N})$, which has the same cardinality as most of our favorite uncountable sets including the real numbers and the (infinite) binary strings.

c. Prove that the number of different chess positions is countable. (A chess position is defined by the locations of pieces on an 8×8 board, where each square on the board can be either empty, or contain a piece from $\{\text{Pawn}, \text{Knight}, \text{Bishop}, \text{Castle}, \text{Queen}, \text{King}\}$ of one of two possible colors.)

Solution: The number of positions is finite, so it is countable.

d. Prove that number of Ziggy Pig ice cream dishes is uncountable. A Ziggy Pig ice cream can contain any number of scoops ($scoops \in \mathbb{N}$), and each scoop can be of any flavor, where distinct flavors are identified by $v \in \mathbb{N}$.

Solution:

We can map $pow(\mathbb{N})$ to the Ziggy Pig ice cream dishes, since each subset of the flavors is a different dish. This proves that it is uncountable. (If you are unfamiliar with the Ziggy Pig, I can only excuse your cultural gap because of your youth, but please aim to correct this travesty by watching "Bill and Ted's Excellent Adventure" over fall break!)

Practice 4 *Vacuous Fish*

Prove that all fish who have eaten Ziggy Pig ice creams (as find in Practice 3) with an infinite number of scoops are Coho Salmon.

Solution: This is vacuously true. Since the amount of sugar in the universe is finite, there exist no Ziggy Pig ice creams with an infinite number of scoops, and the set of all fish who have eaten ice creams with an infinite number of scoops is empty. Any property is true about all elements in an empty set, since there are none of them.

Practice 5 *Induction Practice 1*

Prove by induction that every natural number less than 2^{k+1} can be written as $a_0 \cdot 2^0 + a_1 \cdot 2^1 + a_2 \cdot 2^2 + \cdots + a_k \cdot 2^k$ where all the a_i values are either 0 or 1.

Solution:

For any induction proof, we should start by carefully defining the induction predicate. In this case, it follows directly from the proposition, except we swap k with n (this is just to have the induction predicate take n as its parameter, which is conventional):

$P(n)$ = every natural number less than 2^{n+1} can be written as

$$a_0 \cdot 2^0 + a_1 \cdot 2^1 + a_2 \cdot 2^2 + \cdots + a_n \cdot 2^n$$

where all the a_i values are either 0 or 1.

We want to prove this for all $n \in \mathbb{N}$.

Base Case: $n = 0$.

Since $2^{0+1} = 2$, we need to show that 0 and 1 can both be written as $a_0 \cdot 2^0$: $0 = 0 \cdot 2^0$ ($a_0 = 0$) and $1 = 1 \cdot 2^0$ ($a_0 = 1$).

Inductive Case: $P(n) \implies P(n+1)$.

From $P(n)$, we know all numbers less than 2^{n+1} can be written as

$$a_0 \cdot 2^0 + a_1 \cdot 2^1 + a_2 \cdot 2^2 + \cdots + a_n \cdot 2^n$$

where all the a_i values are either 0 or 1.

To prove $P(n+1)$ we need to show that all numbers less than $2^{(n+1)+1} = 2^{n+2}$ can be written as

$$a_0 \cdot 2^0 + a_1 \cdot 2^1 + a_2 \cdot 2^2 + \cdots + a_n \cdot 2^n + \cdot a_{n+1} \cdot 2^{n+1}$$

where all the a_i values are either 0 or 1.

In comparing to what $P(n)$ means, to show $P(n+1)$ we need to cover the numbers from 0 to $2^{n+2} - 1$, and we have an extra term $a_{n+1} \cdot 2^{n+1}$. If we set $a_{n+1} = 0$, we cover all the numbers from 0 to $2^{n+1} - 1$ because this is the same as $P(n)$.

Each number from 2^{n+1} to $2^{n+2} - 1$ can be written as $m + 2^{n+1}$ where $m \in \{0, \ldots, 2^{n+1} - 1\}$, that is, the numbers covered by $P(n)$ using the terms up to $a_n \cdot 2^n$. So, we cover all the new numbers by setting $a_{n+1} = 1$, and all the old numbers by setting $a_{n+1} = 0$.

Practice 6 *Induction Practice 2*

Prove by induction that every finite non-empty subset of the natural numbers contains a *greatest* element, where an element $x \in S$ is defined as the *greatest* element if $\forall z \in S - \{x\}.\ x > z$.

Solution:

Note that this property may seem trivial, but it is actually quite subtle, and is only true because we limited it to *finite* sets. For example, $\mathbb{N}$ does not have a greatest element (but all subsets of $\mathbb{N}$ do have a least element, which is the well ordering principle).

To prove it, we do induction on the set of the sets. Since we are only dealing with non-empty sets, we are proving the predicate for all elements in $\mathbb{N} - \{0\}$:

$$P(n) = \text{a set of natural numbers of size } n \text{ has a greatest element}$$

Base case: $P(1)$. Every set of size 1 can be written as $\{x\}$ where $x \in \mathbb{N}$. This set has a greatest element, namely x.

Inductive case: $P(n) \implies P(n+1)$.

Every set, T, of size $n+1$ can be written as $T = S \cup \{z\}$ where $|S| = n$ and $z \notin S$ for some $z \in \mathbb{N}$. By $P(n)$, we know S has some greatest element $g \in S$.

We have two cases to consider:

1. $z > g$: The greatest element of T is z.

2. $z < g$: The greatest element of T is g. Since T includes every element of S, we also know $g \in T$.

We know $z \neq g$ since $z \notin S$ and $g \in S$.

So, this covers all possibilities, and in both cases we have a greatest element in T.

Practice 7 *Circuit Evaluation*

In Class 6, we proved that a "good" Boolean circuit always eventually evaluates to a value using the definition of circuit evaluation from Class 6. Prove that a Boolean circuit where there is a cycle on a path between an input and an output will never produce a value for that output.

Solution: We gave an information solution in Class 10. The main idea is to show that a gate's output is undefined until all of its inputs are defined, but if there is a cycle, one of its inputs depends on its output, so will never be defined.