Cardinality of $pow(S)$

Base case: $P(0)$:

$pow(S) = \{\emptyset\}$. $|\,pow(S)| = 1 = 2^0 = 2^{|S|}$.

Inductive case: $\forall m \in \mathbb{N}. P(m) \Rightarrow P(m+1)$.

1. For all sets T where $|T| = m + 1$,
   $\exists S$ where $|S| = m, x \notin S$ such that
   $T = S \cup \{ x \}$
2. $P(m) \Rightarrow |pow(S)| = 2^{|S|}$.
3. Since $pow(T)$ includes all elements of $pow(S)$ as well as those elements with $\{ x \}$ inserted, this means

$$|\,pow(T)\,| = 2 \cdot |pow(S)|$$
$$= 2 \cdot 2^{|S|} = 2^{|S|+1} = 2^{|T|}.$$

QED: For all finite sets S, $pow(S) = 2^{|S|}$.