Midterm

Read this page and fill in your name, pledge, and email ID now.
Do not open past this page until instructed to do so.

Name: ______________________________

UVA Email ID: ______________________________

For this exam, you must **work alone**. You are not permitted to obtain help from people other than asking clarifying questions of the course staff. You are not permitted to provide help to others taking the exam. **You may not use any resources other than your brain and body, the one page of notes you prepared, and a simple writing implement like a pen or pencil.**

Sign below to indicate that you understand these expectations and can be trusted to behave honorably:

Signed: ____

As discussed in the Review Class, our goal is to design an exam that does not incentivize the intellectual dishonesty that is typically incentivized by school assignments and that you are all experts at, as demonstrated by your ability to achieve the level of success needed to be admitted to this selective University. Hence, please keep in mind that the exam will be graded in a way that is intended to *not reward intentionally obfuscated or deceptive answers* — if you do not know how to solve a problem, or get stuck at a step in a proof, it is much better to state that clearly and explain what you know that might be relevant or useful towards solving the problem, than to fabricate an answer that you know is wrong.

We will award fairly generous partial credit for answers that state that you do not know how to solve the asked problem, but either solve an easier one or show something you can do that is related to the given problem. Answers that we believe are deliberately deceptive might will not receive partial credit.

The exam has **8** questions, each of which awards a good answer with 10 points (you can also get up to **20 points for filling in the three blanks** on this cover page well enough so we can read your name and id). For each question, there is ample space provided to hold an excellent answer. If you need more space, you can use the backs of pages, but include clear markings and arrows to indicate the answer that should be graded. We will assume anything not inside and answer box or clearly marked from one, is your scratch work that should not be considered in scoring your answers.

Boolean Circuits and Computing Models

Problem 1 Prove that the gate *AND* is **not** universal, where *AND* has its conventional meaning:

$AND(a, b)$: $AND(1, 1) = 1$, otherwise $AND(a, b) = 0$.

Hint: it would be sufficient to show a function that has one input bit that cannot be implemented by a Boolean circuit using only *AND* gates.

Problem 2 Give a simple description (which could be just the name of a well known function) of the function defined by the code below.

```
def MYSTERY(a, b):
   na = NOT(a)
   nb = NOT(b)
   na_b = AND(na,b)
   nb_a = AND(nb,a)
   return OR(na_b, nb_a)
```

Countability

Problem 3 Suppose S is the set of all (two-footed human) pairs of shoes on Earth. For the ith pair $p_i = (x_i^0, x_i^1)$ let $\overline{p}_i = (x_i^1, x_i^0)$ denote the "wrong" way to wear the shoes (swapped). A configuration for *all* shoes specifies for each pair of shoes whether it is worn correctly or incorrectly. A configuration could be represented by a vector c whose components are indexed by $i \in P$, where P is the set of all two-footed, shoe-wearing humans, and $c_i = p_i$ if person i wears shoes (x_i^0, x_i^1) correctly and $c_i = \overline{p}_i$ if the person i wears the shoes incorrectly. Any configuration c will be a vector of length $|P|$ of these pairs.

Prove that the set $C = \{c \mid c \text{ is a human shoes configuration } \}$ of all such configurations is *countable*.

Problem 4 Suppose we have unlimited time and space, and suppose we have a (self-reproducing) species X such that: at day 1, there is only one x of type X, and at day i, every alive x will reproduce a unique x' (who will be able to reproduce itself on day $i + 1$ and later).

Prove that the set of all creatures of type X that will ever exist is a *countably infinite* set. You should assume these creatures never die or stop reproducing, and time continues forever.

Problem 5 For the creatures in the previous problem, suppose each individual x of creature type X has two feet and a unique pair of shoes. The creatures never die or gain or lose a foot, and time continues forever. As in Problem 3, we call y a configuration for *all* shoes of creatures X, if y determines a correct or an incorrect way of wearing each pair of shoes by each individual $x \in X$. As you hopefully proved in the previous problem, X is a countably infinite set.

Prove that the set $Y = \{y \mid y \text{ is a } X\text{-shoes configuration } \}$ of all such configurations is *uncountable*.

Hint: try to (more) formally write the definition of the configuration for all shoes for X-creatures and connect it to something you already know is uncountable.

Asymptotics

Problem 6 Recall these definitions:

Definition 1 **(O)** A function $f : \mathbb{N} \to \mathbb{R}_+$ is in the set $O(g(n))$ for any function $g : \mathbb{N} \to \mathbb{R}_+$ iff there exist two constants $c \in \mathbb{R}_+, n_0 \in \mathbb{N}$ such that: $\forall n > n_0.\ f(n) \leq cg(n)$.

Definition 2 **(Ω)** A function $f : \mathbb{N} \to \mathbb{R}_+$ is in the set $\Omega(g(n))$, for any function $g : \mathbb{N} \to \mathbb{R}_+$ iff there exist two constants $c \in \mathbb{R}_+, n_0 \in \mathbb{N}$ such that: $\forall n > n_0.\ f(n) \geq cg(n)$.

Definition 3 **(Θ)** A function $f : \mathbb{N} \to \mathbb{R}_+$ is in the set $\Theta(g(n))$, for any function $g : \mathbb{N} \to \mathbb{R}_+$ iff $f(n) \in O(g(n)$ and $f(n) \in \Omega(g(n))$.

Definition 4 **(o)** A function $f(n) : \mathbb{N} \to \mathbb{R}_+$ is in the set $o(g(n))$ for any function $g(n) : \mathbb{N} \to \mathbb{R}_+$ if and only if for every positive constant c, there exists an $n_0 \in \mathbb{N}$ such that $\forall n > n_0.\ f(n) < cg(n)$.

Let $f(n) = 2.001^n$ and $g(n) = 100^{100}n^{100}$, and $h(n) = n^{100} \log n$. For each sub-question, answer if the proposition is *True* or *False*, and support your answer with a convincing argument.

(a) $g \in \Theta(h)$

(b) $g \in o(f)$

(c) $f \in \Omega(f)$

Relation Properties

Problem 7 Suppose there is a bijection b between sets A, B, a total (≥ 1 out) injective (≤ 1 in) function (≤ 1 out) i from set A to set A', and a total surjective (≥ 1 in) function s from set B to set B'.

For each of the statements below, indicate if it is *True* or *False* and provide a brief and convincing justification of your answer.

(a) It must be that $|B'| \leq |A'|$.

(b) There must be a bijection between A and B'.

Induction

Problem 8 Recall the Fibonacci sequence: $F_0 = 0, F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. A pair of positive integers (i, j) are *co-prime*, if they share no factor in common other than the number 1.

Prove that if we ignore F_0, then any two consecutive Fibonacci numbers are co-primes.

Hint: As the title says, use induction! Also, you can use the following fact without proving it: for any $x, y, z \in \mathbb{N}$, if $z = x + y$ and two out of three of $\{x, y, z\}$ are divisible by any integer d, then the third one is (and thus all three are) divisible by d.

Optional Feedback

(Optional) This question is optional and will not affect your grade.

Do you feel your performance on this exam will fairly reflect your understanding of the course material? If not, explain why. (Feel free to alternatively use this space to provide any other comments you want on the exam, the course so far, your thoughts on the CS curriculum, or just draw a picture or your favorite uncountable creature.)

Score:	

End of Exam!

