writers of the past. It also shows that the number of possible truths is infinite.[24]

These ideas had many theological and philosophical consequences and Cantor found that his ideas about the infinite were well received by scholars in these fields. Alas, within mathematics the story was quite different, as we shall see.

chapter five

The Madness of Georg Cantor

'To be listened to is a nearly unique experience for most people. It is enormously stimulating. Man clamors for the freedom to express himself and for knowing that he counts.'

Robert C. Murphy[1]

CANTOR AND SON

'I continued to do arithmetic with my father, passing proudly through fractions to decimals. I eventually arrived at the point where so many cows ate so much grass, and tanks filled with water in so many hours. I found it quite enthralling.'

Agatha Christie[2]

Cantor & Co. was a successful international wholesale business, and as a result young Georg Cantor was one of six children who grew up in comfortable circumstances, attending good private schools in Frankfurt. Georg had many talents and might well have pursued a career as a musician, as did some of his relatives, or as an artist. Yet in his teenage years he became increasing captivated by mathematics, physics, and astronomy. His father, Georg senior, was strongly supportive of all his studies and also imposed his strong religious beliefs in destiny upon his son; some biographers have wondered whether the paternal support was really just a case of the father's own unfulfilled ambitions being pursued through the life of his eldest son. Yet, for all this, Georg junior seems to have survived his life at home, and he graduated from Darmstadt School in 1862, aged seventeen, with high marks, moving first to study mathematics at the Polytechnical Institute in Zurich and then going on to the famed mathematics course at the University of Berlin, the centre of the mathematical world in the mid-nineteenth century. There he encountered great mathematicians like Karl Weierstrass, Sophie Kowalewski, and Ernst Kummer, who followed in the footsteps of men like Bernhard Riemann and Peter Dirichlet. He was also taught by the influential Leopold Kronecker.

Cantor followed the usual route of a young academic of the day, jumping through the hoops laid out for him by completing his degree and then his doctorate in Berlin, before beginning a form of apprenticeship which involved teaching pupils privately at the university in the city of Halle, a medieval city famous for being the birthplace of the great seventeenth-century composer George Frederick Handel. Halle University was an in-between place for a budding mathematician, geographically half-way between the great universities of Berlin and Göttingen; it was the sort of place that you hoped would be a stepping stone to becoming a professor at one of these two famous mathematical centres.

Unfortunately for Cantor, that call never came and he spent the whole of his career in the minor mathematical department in Halle – where there were few visitors and no mathematicians of Cantor's calibre – living comfortably in a big house with his close family following his marriage in 1875 to his sister's friend, Vally Guttman. Things were to become more exciting for Cantor, but not in ways that he could have wished.

THE CHRONICLE OF KRONECKER

'Logic sometimes makes monsters'

Henri Poincaré[3]

The year 1871 was a watershed in Cantor's career as a mathematician. Until that time, his former professor in Zurich, Leopold Kronecker, had been on good terms with him, sympathetic to his work and helpful in getting him established in Halle. He even provided some important mathematical suggestions which helped Cantor to complete some of his first research papers. Then something changed. Cantor began to work on infinities, and in Kronecker's eyes he had suddenly become 'a corrupter of youth'.[4]

Kronecker was the son of a wealthy Prussian businessman and was in no need of a university salary to support his mathematical career

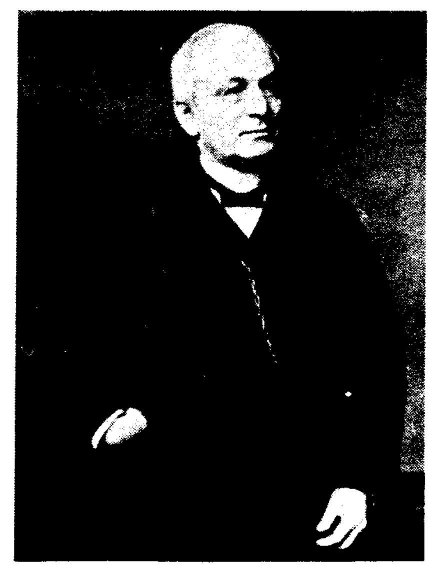


Fig 5.1 *Leopold Kronecker (1823–91).*[5]

(Figure 5.1). He did important work on algebra and number theory in Berlin, but had to spend a period of eleven years away from mathematics while running the family business. Eventually, he returned to become a professor in Berlin in 1882.

The historian of mathematics David Burton writes that

> 'Kronecker was a tiny man, who was increasingly self conscious of his size with age. He took any reference to his height as a slur on his intellectual powers. Making loud voice of his opinions, he was venomous and personal in his attacks on those whose mathematics he disapproved; and his opinions relative to the new theory of infinite sets were ones of ire and indignation . . . Kronecker categorically rejected [Cantor's] ideas [about infinite sets] from the start. He asserted dogmatically, "Definitions must contain the means of reaching a decision in a finite number of steps, and existence proofs must be conducted so that the quantity in question can be calculated with any required degree of accuracy."'[6]

Any discussion of infinite sets was, according to Kronecker, illegitimate since it began with the assumption that infinite sets exist in mathematics.

Kronecker wanted to define mathematics to consist only of those deductions that could be made in a finite number of steps from the natural numbers (1, 2, 3, 4 . . .). This goal is encapsulated in a famous remark he made in a speech: 'God created the natural numbers, and all the rest is the work of man.'

Kronecker was not alone in holding such views, but he was the most influential and vociferous advocate of the mathematical straitjacket called 'finitism'. He believed that we should only do mathematics by building up quantities and arguments in a finite number of steps. Today, this would be classed as the mathematics that a computer could carry out if correctly programmed. We know that this is a small fraction of what is allowed to be mathematics if we do not restrict ourselves to finite step-by-step deductions.

Kronecker would not allow you to assume that something exists if you could not explicitly describe how it could be constructed. Likewise, he would not admit into mathematics those proofs which showed that something must exist without giving the step-by-step recipe for arriving at its construction. In effect, Kronecker believed in a smaller scope for mathematics than did most other mathematicians.

Up until the work of Cantor on infinities, it had been possible to take Gauss's view that infinities in mathematics were always *potential* infinities, and so mention of 'infinity' was just a shorthand for describing a series or a process that had no end: you didn't *do* anything with these infinities. You didn't use them to prove other things were true.

Gauss, the greatest mathematician of the day, had set the tone when he wrote in a letter to his friend Schumacher in 1831 that

> 'I protest against the use of infinite magnitude as something completed, which in mathematics is never permissible. Infinity is merely a façon de parler, the real meaning being a limit which certain ratios approach indefinitely near, while others are permitted to increase without restriction.'

In universities all over the continent of Europe, the division between potential and actual infinities was regarded as crucial, and the general view was that only potential infinities were meaningful.

Despite this current of opinion, most mathematicians held mild views on the issue and rarely encountered a problem where taking a view about finitism really mattered. As a result, most were surprised, and many were irritated, by Kronecker's outspoken finitist views – but the highly-strung, increasingly paranoid Cantor was the most seriously affected by Kronecker's criticisms. All of his work was focused upon defining and manipulating actual infinities and Kronecker characterised this work as a study of things that did not exist, and total 'humbug'![7]

Cantor's hopes of becoming professor of mathematics at the University of Berlin were totally blocked by Kronecker's opposition. Kronecker's influence extended far beyond Berlin, and at Göttingen as well Cantor was repeatedly passed over in favour of seemingly less-distinguished candidates. Kronecker also sat on the editorial boards of journals which delayed or prevented the publication of some of Cantor's work. As a result, Cantor spent his entire professional career, forty-four years, at Halle University, a small college with no mathematical reputation.

Yet Cantor did get his important work published between 1874 and 1884, and it was well known, if occasionally controversial, amongst his young colleagues in Germany at the time – all the more reason for his despair about his lack of advancement. Cantor eventually became so angered by Kronecker's attacks that he wrote directly to the Ministry of Education, hoping to annoy Kronecker by applying for a position vacant in Berlin the following spring. He wrote to his old friend Gösta Mittag-Leffler on 30 December 1883, telling of his desperate measure:

> 'I never thought in the least I would actually come to Berlin . . . since I know that for years Schwarz and Kronecker have intrigued terribly against me, in fear that one day I would come to Berlin, I regarded it as my duty to take the initiative and turn to the Minister himself. I knew precisely the immediate effect this would have: that in fact Kronecker would flare up as if stung by a scorpion, and with his reserve troops would strike up such a howl that Berlin would think it had been transported to the sandy deserts of Africa, with its lions, tigers, and hyenas. It seems that I have actually achieved this goal!'[8]

Kronecker responded the following month by himself writing to Mittag-Leffler (the editor of *Acta Mathematica*) asking if he could publish in his journal a short article setting out his views about certain

mathematical conceptions in which he would show that 'the results of modern . . . set theory [i.e. Cantor's work] are of no real significance'.[9]

Actually, Kronecker had no intention of publishing such a paper, but simply wanted to rattle Cantor into refusing to publish in Mittag-Leffler's journal again in the belief that the editor had betrayed his faith in him by agreeing to publish Kronecker's paper.

At first, however, Cantor was pleased to hear of Kronecker's intention to write a critical article, as it would make Kronecker's opposition public and he would be able to answer it. But then, as Kronecker hoped, Cantor seems to have become suspicious that it would degenerate into personal polemics and told the editor that if the journal published anything critical from Kronecker, he would not support the journal with any of his own work in the future. Kronecker never did send anything to the journal, and the events show something of Cantor's paranoia and despair.

In 1884 Cantor attempted to cool things down by writing directly to Kronecker in a spirit of reconciliation and they had several discussions. However, although Kronecker was outwardly conciliatory, no real peace was made. Cantor concluded there was little hope of success. Indeed, any success Cantor had with others made Kronecker feel even more threatened by Cantor's ideas. Cantor says that, 'It seems to me of no small account that he and his preconceptions have been turned from the offensive *to the defensive* by the success of my work.'[10]

Soon afterwards Mittag-Leffler suggested that one of Cantor's papers should not be published in his journal, saying diplomatically that its insights were 'one hundred years too soon'. This was devastating to Cantor and he never published in the journal again, saying 'I never want to know anything again about *Acta Mathematica*'. (He had also, in 1878, resolved never to publish again in *Crelle's Journal*, another mathematics journal influenced by Kronecker.) As a result, by 1885 he had decided to give up mathematics entirely (Figure 5.2).

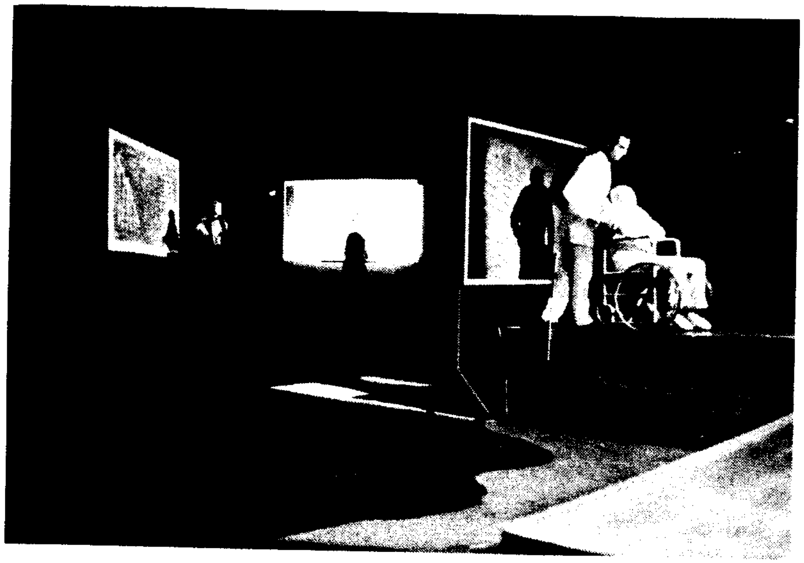
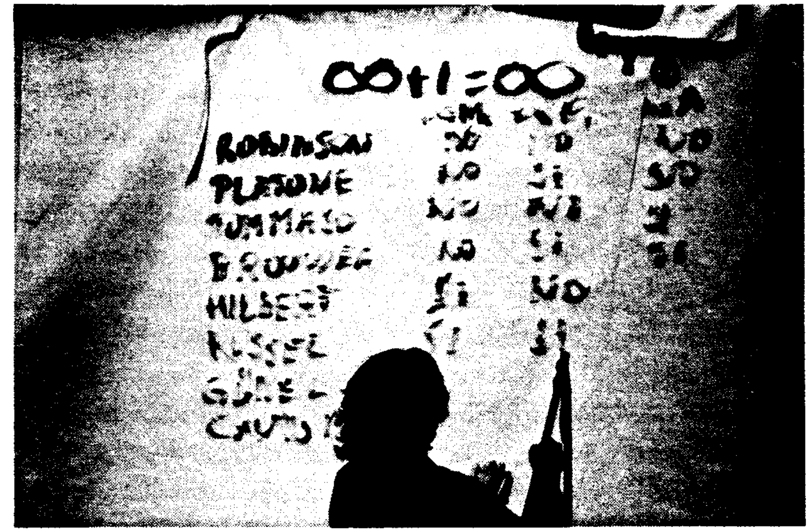


Fig 5.2 *Pictures of Cantor's struggle with mathematics and mathematicians from the Milan production of* Infinities.

Cantor's belief that he was being persecuted by Kronecker because of his mathematical views led to a complete nervous breakdown in 1884. He made a recovery one month later, but subsequently his life was punctuated by bouts of depressive illness which forced him to stay for periods in the clinic in Halle. In the intervals when his mind was clear, he spent a lot of time on studies of the ancient history of counting systems, theology, and history. It was not until the 1900s, when he had finished his research, that Cantor's work started to receive international recognition, with many prizes and honorary degrees being awarded to him. However, this recognition came mainly from outside Germany and Cantor complained, in 1908, of the German mathematicians 'who do not seem to know me, although I have lived and worked among them for fifty-two years'.

Ultimately, as we will see, these events and stresses tipped him into depression and undermined his belief in the worth of his own or any mathematical research. He attempted to transfer from the mathe-



matics department to the philosophy department at his university – a request that was refused. Yet, the university bent over backwards to give him time to rehabilitate himself, and hired temporary lecturers to deputise for him during his periods of illness and absence. To escape these periods of depression Cantor began contemplating the theological implications of his work on the infinite. Its reception by theologians was unexpected.

CANTOR, GOD, AND INFINITY – THE TRINITY WITH AFFINITY

'I entertain no doubts as to the truth of the transfinities, which I have recognised with God's help and which, in their diversity, I have studied for more than twenty years; every year, and almost every day brings me further in this science.'

Georg Cantor[11]

In 1885 Cantor put mathematics to one side and started to correspond with theologians and other intellectuals about infinity. Always someone of strong religious faith, and strongly influenced by his father's forceful beliefs, his attitude towards his work on infinity began to shift in an unusual way. He started to tell his friends that he had not been the inventor of the ideas about infinity that he had published. He was merely a mouthpiece, inspired by God to communicate parts of the mind of God to everyone else. This increased his belief in the truth of his work on infinity, for in his mind it had risen to the elevated status of revealed truth.

Cantor had changed direction at just the right time. The mathematical world in his vicinity may have been under the conservative influence of Kronecker's outspoken views, but when Leo XIII ascended to the papacy in 1878 he brought a liberalisation of the Church's attitudes in many areas. He sought to reconcile science and religion by offering a more enlightened lead from Rome.

This was good news for one Constantin Gutberlet, a priest, philosopher and theologian, and one of Germany's leading neo-Thomists. Gutberlet believed controversially that the human mind could grasp actual infinities and talk meaningfully about them. As a result, he had come under attack from Catholic theologians, but had responded by seizing upon Cantor's mathematical work to argue that it provided clear evidence that the human mind could contemplate the actual infinite. Moreover, if it did so, it would get closer to the true nature of the Divine. The collection of divine thoughts in the mind of an unchanging God, he argued, must comprise a complete and infinite set. This was for him evidence that Cantor's infinities actually existed, and to deny it would require you to give up the infinite and absolute mind of God. High stakes indeed.

Gutberlet's approach is reminiscent of the way in which Euclid's geometry had played an important role in supporting claims that the human mind could have access to matters of ultimate truth. If theologians were challenged by sceptics who argued that ultimate truth was

something that transcended the human mind, they could point to Euclid as an example of part of the ultimate truth about the Universe that we have found. In the nineteenth century there would be radical changes to our view of mathematical structures like Euclid's geometry. No longer would it be possible to argue that Euclid's geometry was the one and only logically possible geometry and therefore tells how the world must necessarily be. It was recognised that there can exist other non-Euclidean geometries – infinitely many of them – all logically self-consistent. The fact that they exist mathematically by virtue of being logically self-consistent does not mean that they must exist in physical reality though.

Gutberlet wrote about the vital theological importance of Cantor's work, and entered into correspondence with him over the question of the absolute infinity of God's existence. Cantor was extremely interested in the theological consequences of his ideas, and argued that the higher infinities he had found increased the extent of God's dominion for they had no upper bound: there was no 'biggest' infinity. His never-ending tower of infinities provided a simple answer to the challenge that Gutberlet was facing, that understanding and codifying infinity was reducing the status of God. This might well have been worrying to some, had there been a biggest infinity.

Cantor believed that he could use his knowledge to prevent the Church making grave errors about its doctrines concerning infinity. He thought it was a mission to which he had been called. He declared in a letter to a friend, in 1896, that

> 'From me, Christian philosophy will be offered for the first time the true theory of the infinite.'[12]

He also said,

> 'But now I thank God, the all-wise and the all-good, that He always denied me the fulfilment of this wish [for a position at university either in Göttingen or Berlin], for He

> thereby constrained me, through a deeper penetration into theology, to serve Him and his Holy Roman Catholic Church better than I have been able with my exclusive preoccupation with mathematics.'[13]

Many have felt that Cantor was signalling his despair with all that had gone before and was just turning to a less demanding and controversial activity, away from Kronecker and the rivalries of other mathematicians. However, he interpreted his growing liking for theology and philosophy and his disaffection with mathematics as the work of God. He saw himself as a servant of God who had been given the talent for mathematics in order to be of service to the Church.

He gave up contact with his mathematical friends and was happy about his contacts with Church theologians and philosophers who were interested in his work and thought it significant. Religion renewed his self-confidence and convinced him that his work was important after all, despite the opposition of so many mathematicians. In 1887, Cantor wrote to his colleague Heman of his confidence that he could answer any criticism and overcome any opposition:

> 'My theory stands as firm as a rock; every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides for many years; because I have examined all objections which have ever been made against the infinite numbers; and above all, because I have followed its roots, so to speak, to the first infallible cause of all created things.'[14]

Georg Cantor was very interested in how mathematics might reveal the existence of God. In letters to Cardinal Franzelin, he indicated that the infinite, or the 'Absolute', belonged uniquely to God. He believed that it was God who ensured that the hierarchy of transfinite numbers existed, stretching beyond the simplest countable infinities,

increasing without limit. Because the largest of these could never be captured by a single formula – from any infinite set it was always possible to make an infinitely larger one – Cantor regarded the transfinite numbers as ascending directly to the Absolute, to the 'true infinity' whose magnitude was an absolute maximum that was incomprehensible to mere human understanding. The Absolute Infinite was beyond human determination, since once it was determined, the Absolute would no longer be regarded as infinite, because it would then necessarily be finite by definition – once determined it could be added and subtracted and manipulated or infinitely increased, just like the lesser infinities.

Thus Cantor seems to think of Absolute Infinity in the way that Archbishop Anselm thought of God in his famous 'ontological' proof of the existence of God, as being that above which no greater could be conceived.

What did Cantor's colleagues think about his ideas on God and infinity? Constantin Gutberlet had studied under Franzelin. He corresponded with Cantor and took his ideas very seriously. At first he was worried that Cantor's work on mathematical infinity challenged the unique, 'absolute infinity' of God's existence. However, Cantor assured him that instead of diminishing the extent of God's dominion, the transfinite numbers actually made it greater. After talking to Gutberlet, Cantor became even more interested in the theological aspects of his own theory on transfinite numbers.

Furthermore, Gutberlet argued that since the mind of God was unchanging, the collection of Divine thoughts must comprise an absolute, infinite, complete closed set, and offered this as direct evidence for the reality of concepts like Cantor's transfinite numbers. Like Pythagoras and Plato, Cantor believed that the numbers (particularly his transfinite numbers) were externally existing realities in the mind of God. They were discovered. They followed God-given laws, and Cantor believed it was possible to prove their existence from God's perfection and power. Indeed, Cantor said, it would have diminished God's power had God only created finite numbers.

Ironically, Cantor's love of the infinite had a distinctly anti-Pythagorean flavour. Pythagoras believed infinity was the destroyer in the Universe, the malevolent annihilator of worlds. If mathematics were a war, then the struggle was between the finite and the infinite. The Pythagoreans became obsessed with the negative aspects of infinity. They believed that the whole numbers closest to one (and therefore the 'most' finite in some sense of being farthest from the infinite) were the most pure of all numbers.

ALL'S SAD THAT ENDS BAD

'Behold the heaven of heavens cannot contain Thee'

The book of Chronicles[15]

Leopold Kronecker died in 1891 without ever becoming involved in a public criticism of Cantor's work. After 1895, a few of Kronecker's old allies opposed Cantor's ideas but, increasingly, the younger mathematicians supported Cantor and the dispute over finitism just faded away.[16] Cantor, however, never regained his mathematical powers and his decline had a terrible inevitability about it.

As we have seen, he had suffered his first breakdown in May 1884, just after his thirty-ninth birthday. He returned to doing mathematics in the autumn, but his interests had changed. He spent a lot of time working on Elizabethan history (trying to prove that Francis Bacon wrote Shakespeare's plays!), and early theology.

Eventually he suffered further breakdowns, and was in hospital for part of 1899 because of mental instability. He applied for leave of absence from teaching at Halle and wrote to the Ministry of Culture saying he wanted to leave his professorship. If they would pay him the same salary, he would be happy to take a quiet position in a library somewhere. He wanted to break away from maths and stressed his knowledge of history and theology. He even threatened to apply to join the Russian diplomatic service. All this came to nothing.

In December 1899, while he was out giving a lecture in Leipzig about the Bacon-Shakespeare authorship issue, his youngest son, Rudolf, died suddenly just before his thirteenth birthday. Rudolf, although always frail and in poor health, had been a gifted musician, just as his father had been as a child before he gave up music for mathematics. Despite this cruel blow, Cantor managed to remain of sound mind for three years, but was back in hospital, relieved of his teaching duties again, in the winter of 1902–3. Some of his work was questioned in a public conference in 1904 and this agitated him greatly. He was in hospital during the winter of 1904–5, in 1907–8 and 1911–12. In 1915 an international meeting was planned to celebrate his seventieth birthday, but the war prevented all but a few close German friends from attending. He was admitted to the Halle clinic for the final time on 11 May 1917. He didn't return home. In wartime rationing conditions, food was scarce and he lost weight steadily. He died of heart failure on 6 January 1918, twenty-seven years after Kronecker. At the end of the game, the pawn and the king go back in the same box.