Exam 1

Read this page and fill in your name, pledge, and email ID now.
Do not open past this page until instructed to do so.

Name: _____

UVA Email ID: _____

For this exam, you must **work alone**. You are not permitted to obtain help from people other than asking clarifying questions of the course staff. You are not permitted to provide help to others taking the exam. **You may not use any resources other than your brain and body, the one page of notes you prepared, and a simple writing implement like a pen or pencil.**

Sign below to indicate that you understand these expectations and can be trusted to behave honorably:

Signed: _____

As discussed in the Review Class, our goal is to design an exam that does not incentivize the intellectual dishonesty that is typically incentivized by school assignments and that you are all experts at, as demonstrated by your ability to achieve the level of success needed in high school to be admitted to the University. Hence, please keep in mind that the exam will be graded in a way that we hope will not reward intentionally obfuscated or deceptive answers — if you do not know how to solve a problem, or get stuck at a step in a proof, it is much better to state that clearly and explain what you know that might be relevant or useful towards solving the problem, than to fabricate an answer that you know is wrong.

Answers that we believe are deliberately deceptive will receive negative scores (worse than that 0 that a blank answer receives for any question), although fairly generous partial credit will be awarded for answers that state that you do not know how to solve the asked problem, but either solve an easier one or show something you can do that is related to the given problem.

To that goal, we have also included one *impossible* problem in the exam: one problem on the exam asks you to prove something that is untrue. You receive full credit for this problem by just writing "Impossible!" as your answer. If you want to include an explanation you can, but it is only (possibly) beneficial if you mark a problem that we think is answerable as impossible.

The exam has **9** questions, each of which awards a good answer with 10 points (you can also get up to **15 points for filling in the three blanks** on this cover page well enough so we can read your name and id). For each question, there is ample space provided to hold an excellent answer. If you need more space, you can use the backs of pages, but include clear markings and arrows to indicate the answer that should be graded. We will assume anything not inside and answer box or clearly marked from one, is your scratch work that should not be considered in scoring your answers.

Boolean Circuits

For these questions, we assume the following logical functions with their standard meanings:

NOT(a): *NOT*$(0) = 1$, *NOT*$(1) = 1$.
OR(a, b): *OR*$(0, 0) = 0$, otherwise *OR*$(a, b) = 1$.
AND(a, b): *AND*$(1, 1) = 1$, otherwise *AND*$(a, b) = 0$.

1. Give a simple description (which could be just the name of a well known function) of the function defined by the code below:

```
def MYSTERY(a, b):
    v1 = NAND(a, b)
    return NAND(v1, v1)
```

2. Define *XOR* using only *NOT, OR* and *AND,* where *XOR*$(0, 0) =$ *XOR*$(1, 1) = 0$ and *XOR*$(0, 1) =$ *XOR*$(1, 0) = 1$.

Countability

For these problems, you may use any results that were proven in class or on a problem set in your proof (without needing to prove them).

3. Prove that the set of all fish in the sea is *countable*. (For purposes of this question, you can assume the "sea" in question is the Mediterranean Sea, and *fish* has its conventional meaning.)

4. Prove that the set of the even natural numbers (i.e., $\{0, 2, 4, 6, \ldots\}$) is *countably infinite*.

5. Prove that the set of all directed graphs (as defined below) is *uncountable*.

Definition 1 (Directed Graph) A *directed graph* $G = (V, E)$ consists of a (possibly infinite) set V of vertices and a (possibly infinite) set E of edges. Every edge is an ordered pair of two distinct elements of V.

Proofs with Definitions

Here we define the Counting Numbers, similarly to the definition of Natural Numbers you saw on Problem Set 1:

Definition 2 (Counting Numbers) We define the *Counting Numbers* as:

1. **1** is a Counting Number.
2. If n is a Counting Number, **S**(n) is a Counting Number.

6. Prove that the cardinality of the set of all *Counting Numbers* (as defined above) is *countably infinite*.

Computing Models

Reminder: make sure you have read the grey box on the cover page.

Recall from Class 5 and the book that two computing models are *equivalent* in power if every computation that can be defined using the first model can also be defined using the second model, and every computation that can be defined using the second model can also be defined using the first model.

7. Prove that Boolean circuits using the gateset $\{MAJ, NOT, ZERO\}$ is equivalent to Boolean circuits using the gateset $\{MAJ, NOT, ONE\}$. ($ZERO$ is the constant gate that outputs `0` for any input, and ONE is the constant gate that outputs `1` for any input. MAJ is the majority of three inputs gate you are familiar with from PS3.)

8. Prove that Boolean circuits using the gateset $\{MAJ, NOT\}$ is equivalent to Boolean circuits using the gateset $\{NAND, XOR\}$.

9. Prove that *AON-CIRC*, straightline programs composed of *AND*, *OR*, and *NOT* operations is equivalent to *MOP-CIRC*, straightline programs composed of the plus, multiply, and constant one operations defined by:

```
def PLUS(a, b):
    return (a + b) % 2


def MULT(a, b):
    return (a * b) % 2


def ONE(a, b):
    return 1
```

The `%` operator is modulo (remainder after division). So, for example `(0 + 1) % 2 = 1`, `(1 + 1) % 2 = 0`, and `(1 * 1) % 2 = 1`.

Optional Feedback

This question is optional and will not negatively affect your grade. (In rare circumstances, it might increase your grade, or lead us to provide you with some additional opportunity to do so.)

Do you feel your performance on this exam will fairly reflect your understanding of the course material so far? If not, explain why. (Feel free to provide any other comments you want on the exam, the course so far, your hopes for the rest of the course here, or just draw a picture.)

Score:	

End of Exam 1